FLUID FLOWS AROUND RECTANGULAR STEPS AND BOULDERS

Paul Christodoulides[1*], Eric Vacelet[2] & Frederic Dias[2,3]
[1]*Faculty of Engineering and Technology, Cyprus University of Technology, Limassol, Cyprus*
[2]*Université Paris-Saclay, ENS Paris-Saclay, Gif-sur-Yvette,Centre Borelli, France*
[3]*School of Mathematical Sciences, University College Dublin, Dublin, Ireland*

Summary Fluid flows around obstacles constitute a classic problem in fluid dynamics. Steady two-dimensional fluid flows over an obstacle can be solved using complex variable methods. In particular, the impact of free-surface flows hitting a vertical wall of finite extent is studied here, for an inviscid and incompressible fluid; the flow is steady and irrotational. The flows are uniform far upstream and far downstram with constant but different velocities and depths. The solution of such problems depends on the depth ratios and on the dimensionless upstream and downstream Froude numbers. Various numerical solutions of the problem are presented. Relevant pressure and forces are also addressed with regard to the "boulder" problem

INTRODUCTION

Fluid flow around an obstacle constitutes a classical problem in applied mathematics and fluid dynamics. Such flows include flows past semi-circles, semi-ellipses, ramps, triangles, rectangles and other, which may be considered as two-dimensional approximations of realistic regimes in nature [1],[2],[3]. A limiting example of an obstacle is a vertical wall that forces the fluid to move upward.

Using the dimensionless Froude number, the solutions to the steady two-dimensional flows of such problems can be classified into 11 basic steady flow types, namely 2 types of supercritical, 4 types of critical and 3 types of subcritical with or without waves, and 2 types of hydraulic falls [4]. One may be interested in the impact pressure against the vertical wall [5]. The wave may or may not be breaking, and this fundamentally changes the types of force exerted on the wall.

In the current study we are interested in the impact of a flow hitting a vertical wall of finite extent. This can be a simplification of a rectangular boulder lodged against a platform step. This is actually a problem considered in [6], where no free surface was considered nor any influence by gravity.

METHODOLOGY

Problem formulation

As shown in Figure 1, the fluid moves over a finite vertical wall of height W. The fluid is inviscid and incompressible and the flow is assumed to be steady and irrotational. Far upstream the flow is uniform of constant velocity U_1 and depth $W + H_1$, H_1 being the vertical coordinate of the free surface. Far downstream the flow is also uniform of constant velocity U_2 and depth H_2. The incoming and outcoming mass flux is defined as

$$Q = U_1(W + H_1) = U_2 H_2. \tag{1}$$

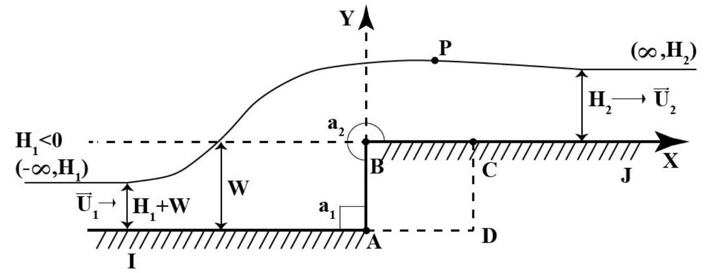

Figure 1. Sketch of the flow and of the coordinates. Special points are labeled on the boundary.

It turns out that the solution of such a problem depends on the ratios W/H_2 and H_1/H_2 and on the dimensionless upstream and downstream Froude numbers, respectively,

$$F_1 = \frac{U_1}{\sqrt{g(H_1 + W)}}, \qquad F_2 = \frac{U_2}{\sqrt{gH_2}}, \tag{2}$$

where g is the acceleration due to gravity. The flows considered here are supercritical both upstream and downstream, i.e., $F_1 > 1$, $F_2 > 1$ [2]. Note that the two Froude numbers are interlinked through equation (1). The system is governed by Bernoulli's equation that, assuming that the pressure is constant on all free surfaces, yields (upon taking H_2 and U_2 as the unit length and velocity respectively) in dimensionless form (u and v are the x- and y-components of the fluid velocity),

$$\frac{1}{2}(u^2 + v^2) + \frac{1}{F_2^2} y = \frac{1}{2} + \frac{1}{F_2^2}. \tag{3}$$

The problem in question can be solved with the use of conformal mappings, using the velocity potential $\phi(x, y)$ (see irrotational flow) and streamfunction $\psi(x, y)$ respectively and defining complex variables $z = x + \mathrm{i}y$ and $f = \phi + \mathrm{i}\psi$. The (conformal) transformation from the [f]-plane to the [t]-plane (upper half of unit circle - details omitted here) can be written in differential form as (where $t = e^{\mathrm{i}\sigma}$, $\sigma \in [0, \pi]$, denote the points on the free surfaces in the [t]-plane)

* Corresponding author. Email: paul.christodoulides@cut.ac.cy

$$\frac{df}{dt} = \frac{4}{\pi}\frac{1}{1-t^2}. \tag{4}$$

The problem reduces to finding hodograph variable $\zeta(z) = df/dz$ as an analytic function of t, satisfying Bernoulli's equation (3) and a kinematic boundary condition. An analysis of singularities at points A and B (see Figure 1), leads to the relation (for an analytic function $\Omega(t)$, λ_1, λ_2 related to F_1, F_2 at infinity respectively, K some constant)

$$\zeta(t) = \frac{(t-t_A)^{1/2}}{t^{1/2}} e^{\Omega(t)}, \qquad \Omega(t) = K(1-t)^{2\lambda_2/\pi}(1+t)^{2\lambda_1/\pi} + \sum_{n=0}^{\infty} a_n t^n. \tag{5}$$

Numerical method

The coefficients a_n in the power series above can be determined by collocation, upon applying series truncation after $N-2$ terms and introducing $N-2$ mesh points $\sigma_M = \pi(M-0.5)/(N-2)$ on the free surfaces. The values of x and y in Bernoulli's equation (2), are computed through integration of relation: $\frac{dz}{dt} = \frac{1}{\zeta}\frac{df}{dt}$.

Substituting the expressions of x, y and ζ into equation (3) at mesh points σ_M yields $N-2$ nonlinear algebraic equations for the N unknowns a_0, … a_{N-3}, K and λ_1,. The last two equations are obtained by satisfying upstream and downstream conditions (see equation (1)). Note that λ_1, λ_2 are interlinked through equation (1).

This system of N nonlinear equations in N unknowns is solved by Newton's method for given values of Froude number F_2 and t_A, using MATLAB. Most computations were performed with $N = 201$, except when higher regional accuracy was required, $N = 401$ was used.

RESULTS AND DISCUSSION

Exponential decay of the solution

As a first step, the exponential decay of the solution at $x \to -\infty$ was considered, following the procedure described above. Using $N = 401$, for $F_2^2 = 2$, $t_A = -0.95$, the flow free surface profile is shown in Figure 2; note that $\lambda_1 = 1.519$, $F_1^2 = 12.654$.

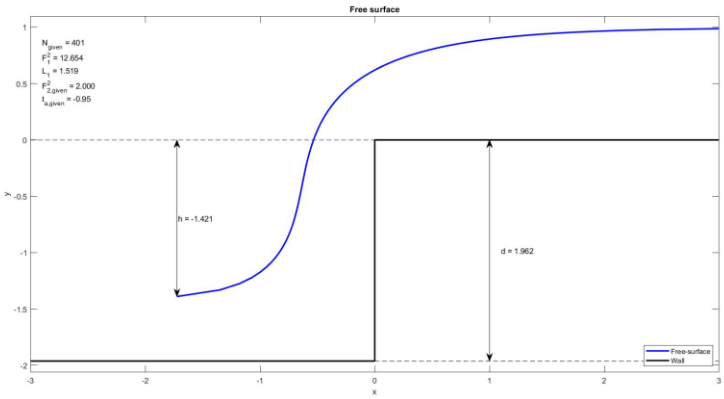

Figure 2. Free surface with exponential decay at $x \to -\infty$ for $N = 401$, $F_2^2 = 2$, $t_A = -0.95$.

For cases where exponential decay is not considered, the system is simpler as the corresponding equation (5) for $\Omega(t)$ should not contain λ_1. Accurate numerical profiles can be obtained using $N = 201$.

Pressure and forces

The pressure in the fluid can be determined through Bernoulli's equation,

$$\frac{1}{2}(u^2 + v^2) + gy + \frac{p}{\rho} = \text{constant}. \tag{6}$$

Following [6], one can compute the pressure on a boulder, described here as a two-dimensional predetermined rectangular block. It turns out that the pressure is negative (see equation (3)). Subsequently, as the pressure is integrable, one can further determine the forces exerted on the boulder.

Based on the obtained results, limiting and other relevant flows can also be addressed. Some notable examples open channel flows with submerged obstructions as well as well as a flow hitting a wall, with the fluid splitting into two parts.

References

[1] King A. C., Bloor M. L. G. Free-surface flow over a step. *Journal of Fluid Mechanics* **182**:193-208, 1987.

[2] Dias F., Vanden-Broeck J.-M. Open channel flows with submerged obstructions. *Journal of Fluid Mechanics* **206**:155-170, 1989.

[3] Binder B.J., Dias F., Vanden-Broeck J.-M. Free-surface flow past arbitrary topography and an inverse approach for wave-free solutions. *Theoretical and Computational Fluid Dynamics* **20**(3):125-144, 2006.

[4] Binder B. J., Blyth M. G., McCue S. W. Free-surface flow past arbitrary topography and an inverse approach for wave-free solutions. *IMA Journal of Applied Mathematics* **78**(4):685-696, 2013.

[5] Dias F., Ghidaglia J.-M. Slamming: Recent progress in the evaluation of impact pressures: *Annual Review of Fluid Mechanics* **50**:243-273, 2018.

[6] Herterich J., Dias F. Potential flow over a submerged rectangular obstacle: Consequences for initiation of boulder motion. *European Journal of Applied Mathematics* 31:646-681, 2020.