Stability of interfacial waves between two bounded fluids

Paul Christodoulides

Department of Mathematics, Monash University
Clayton, Victoria 3168, Australia

Abstract

Two-dimensional periodic gravity waves at the interface between two bounded fluids of different densities are considered. Based on the Hamiltonian structure of the problem, the relation between wave frequency and wave amplitude is obtained through a weakly nonlinear analysis. All classes of time- and space-periodic waves are studied: traveling and standing waves as well as a degenerate class of mixed waves. As opposed to water waves, mixed interfacial waves exist even in the absence of capillarity. The stability of traveling and standing waves with respect to three-dimensional modulations is then studied. By using the method of multiple scales, Davey–Stewartson-type equations are obtained. A detailed stability analysis is performed and reveals that while for longitudinal and oblique modulations standing and traveling waves have the same stability behavior, for transverse modulations standing waves are less stable than traveling waves.

1. Introduction

The problem of periodic waves at the interface between two inviscid fluids of unequal depths and different densities in the presence of gravity is considered. The goal is to study the bifurcations and modulational stability of *all* classes of periodic interfacial waves: traveling waves (TWs), standing waves (SWs) and mixed waves (MWs). This kind of waves may find applications in the ocean or even in lakes, eventhough it must be said that this system may be subject to other instabilities of the Kelvin–Helmholtz type when the wave amplitudes are large enough and may even be affected by viscosity. However the simplified system studied here is itself very useful towards understanding basic features of interfacial waves.

Unless otherwise stated, the references below deal with interfacial waves. Among numerous contributions regarding TWs are Thorpe's (1968*a*) theoretical and experimental investigation of gravity waves in finite depth, Tsuji and Nagata's (1973) formal study of deep internal waves through a fifth-order expansion, and Holyer's (1979) numerical study of gravity waves. Nayfeh (1976) studied the stability of TWs with respect to small two-dimensional modulations when both layers have infinite depth, while Tanaka (1982) and Grimshaw and Pullin (1985) treated the finite-depth case.

The literature concerned with SWs is quite limited. One may note Thorpe's (1968*b*) theoretical and experimental results for finite-depth gravity waves as well as Dias and Bridges's (1994) analytical study. To our knowledge no stability results exist for interfacial SWs, which is one of the main motivations of the current study.

The third class of periodic waves considered here is the class of mixed waves. For the linearized problem, mixed waves are the superposition of a left TW and of a right TW with different amplitudes. However mixed waves do not persist when nonlinear terms are considered, except in degenerate cases. Dias and Bridges (1994) showed that mixed waves exist in infinite depth when a certain relation between density ratio and interfacial tension is satisfied. Here we show that, in finite depth, mixed waves can occur even if interfacial tension is neglected.

The present study is divided into two parts. In the first part (§ § 2–3), the objective is to obtain the so-called *normal form* of the problem, which gives a relation between the angular frequency ω and the wave amplitude for fixed wavenumber k and, consequently, gives important information about the behavior and properties thereof. In the second part (§ § 4–5), we derive Davey–Stewartson-type equations, through which a stability analysis of interfacial waves with respect to small modulations is performed.

The study in the first part is based on the Hamiltonian structure of the problem, which has been given (for interfacial waves) by Benjamin and Bridges (1991): The formulation of the problem is recalled in § 2. The equivalent Lagrangian formulation is provided and a fourth-order truncation of the Lagrangian $\mathcal{L}$ is formally derived. The general form of Hamiltonian $\mathcal{H}$ or Lagrangian $\mathcal{L}$ may be obtained simply by considering the symmetries of the problem. However, the computation of the values of the coefficients in $\mathcal{L}$ in terms of the physical parameters requires tedious work. The coefficients are computed in the general case of finite depths through a projection method similar to Whitham's (1974) averaging method. Note, however, that in this case the method is only formally constructed and its convergence is not assured. In § 3, considering the first variations of $\mathcal{L}$ yields the desired normal form (truncated at order 2) which reveals the presence of three classes of waves: *TW*s, *SW*s and degenerate *MW*s. As said above, the new feature is that while for water waves these *MW*s were only possible in the presence of capillarity, they exist here even for pure gravity waves.

In the second part (§ § 4–5), our aim is to study the stability of two-dimensional interfacial waves with respect to small *three*-dimensional modulations, in the spirit of Davey and Stewartson's (1974) analysis. The first stability results on interfacial waves are due to Nayfeh (1976) who conducted a nonlinear stability analysis on two-dimensional capillary–gravity waves between two unbounded fluids, using multiple scales. Tanaka (1982) and Grimshaw and Pullin (1985) studied the stability of gravity interfacial waves with respect to three-dimensional modulations. Pullin and Grimshaw (1985, 1986) extended this study to finite-amplitude waves, with or without the presence of a basic current. However, no work has dealt so far with the stability of interfacial *SW*s.

The present analysis treats the stability of both *TW*s and *SW*s with respect to three-dimensional modulations. This was first done for surface waves by Pierce and Knobloch (1994). They found that *SW*s may be unstable for transverse modulations even if they are stable with respect to longitudinal modulations. As in their case, the multiple scales expansion method is used here as framework. To handle *SW*s, one may think of them as a superposition of two counter-propagating *TW*s of equal amplitudes. Therefore, asymptotic expansions with respect to a small ordering parameter ε will be considered both in $e^{i(kx-\omega t)}$ and in $e^{-i(kx-\omega t)}$, where x is the spatial coordinate along which the wave of wavenumber k and angular frequency ω propagates in time t. The analysis in § 4 leads to a system of three partial differential equations (see (33)–(34)) analogous to the two Davey–Stewartson equations (the third one here is due to the addition of a second fluid-layer). Equation (34) is a nonlinear Schrödinger-type equation, which is different for *TW*s and *SW*s. The reason behind this is that while *TW*s depend on time at order ε^2 (since at order ε they may be transformed by $x \mapsto x - c_g t$), *SW*s depend on t at order ε which yields additional terms in (34). Based on (33)–(34), a stability analysis in § 5 with respect to both longitudinal and transverse small modulations provides interesting new results for both *SW*s and *TW*s. A first observation is that when only longitudinal modulations (i.e. modulations in x) are present, *SW*s have the same stability properties as *TW*s. However, when only transverse modulations are considered, *SW*s have different properties and are in fact less stable than *TW*s. As for the role of the density ratio, one may say that it is in general stabilizing, especially for *TW*s.

2. Hamiltonian formulation and weakly nonlinear analysis

The geometry of the problem is presented in Fig. 1. A fluid of density ρ' lies on top of a heavier fluid of density ρ. The plane $z = 0$ represents the interface at rest. When in motion, the interface is described by $z = \eta(x, y, t)$, where x and y denote the horizontal coordinates. The lower fluid is bounded below by a horizontal plane at $z = -\delta$, while the upper fluid is bounded above by a horizontal plane at $z = \delta'$. Both fluids are inviscid and incompressible. The flows in each fluid are assumed to be irrotational. Therefore, velocity potentials are introduced in each fluid. The governing equations are given by

$$\nabla \cdot \vec{u} = \nabla^2 \phi = 0 \,, \qquad \nabla \cdot \vec{u}' = \nabla^2 \phi' = 0 \,, \tag{1}$$

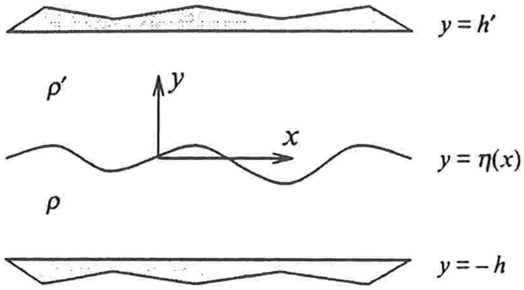

Figure 1. Geometry of the two-layer fluid problem.

where $\vec{u}$ and $\vec{u}'$ are the velocities of the lower and upper fluids respectively, generated by corresponding potentials ϕ and ϕ', subject to conditions

$$\phi_z = 0 \quad \text{at} \quad z = -\delta\,, \qquad \phi'_z = 0 \quad \text{at} \quad z = \delta'\,. \tag{2}$$

At the interface $z = \eta(x, y, t)$, the two kinematic conditions are given by

$$\eta_t - \Phi_{(z)} + \eta_x \Phi_{(x)} + \eta_y \Phi_{(y)} = \eta_t - \Phi'_{(z)} + \eta_x \Phi'_{(x)} + \eta_y \Phi'_{(y)} = 0\,, \tag{3}$$

and the dynamic condition (obtained from Bernoulli's equation) by

$$\rho\left(\Phi_{(t)} + \tfrac{1}{2}|\mathbf{u}|^2\right) - \rho'\left(\Phi'_{(t)} + \tfrac{1}{2}|\mathbf{u}'|^2\right) + g(\rho - \rho')\eta = 0\,, \tag{4}$$

where g is the acceleration due to gravity, $\Phi_{(*)} = \phi_{(*)}(x, y, \eta, t)$, $\Phi'_{(*)} = \phi'_{(*)}(x, y, \eta, t)$, $\mathbf{u} = \vec{u}(x, y, \eta, t)$, $\mathbf{u}' = \vec{u}'(x, y, \eta, t)$ and subscripts in parentheses denote evaluation of derivatives at the interface. Note that the horizontal coordinate y has been included in the formulation above because it will be considered in the stability analysis (see § 4) of the two-dimensional waves obtained in § 3.

The above problem may be represented as a Hamiltonian system that we briefly recall below for the case of two-dimensional periodic solutions in space (of wavelnumber k) and in time (of frequency ω). For a detailed analysis see Benjamin and Bridges (1991). The evolution equations are

$$\eta_t = \frac{\Delta\mathcal{H}(\zeta,\eta)}{\Delta\zeta}\,, \qquad \zeta_t = -\frac{\Delta\mathcal{H}(\zeta,\eta)}{\Delta\eta}\,, \tag{5}$$

where canonical variable ζ is equal to $\rho\Phi - \rho'\Phi'$ and Δ denote variational derivatives. The Hamiltonian function $\mathcal{H}$ is the sum of $K + V$, where K and V are respectively the kinetic and potential energies, given by

$$K(\phi,\, \phi',\, \eta) \;=\; \int_0^{\frac{2\pi}{k}} \left[\int_{-\delta}^{\eta} \tfrac{1}{2}\rho|\nabla\phi|^2\, \mathrm{d}z + \int_{\eta}^{+\delta'} \tfrac{1}{2}\rho'|\nabla\phi'|^2\, \mathrm{d}z\right]\,, \tag{6}$$

$$V(\eta) \;=\; \int_0^{\frac{2\pi}{k}} \tfrac{1}{2}(\rho - \rho')g\eta^2\, \mathrm{d}x\,. \tag{7}$$

In (6) K is given as a function of ϕ and ϕ'. In fact, it can be proved that ϕ and ϕ' only appear in the combination ζ through the calculus of variations. That is, $K(\zeta,\, \eta)$ is obtained as the minimum of $K(\phi,\, \phi',\, \eta)$ with η fixed on the constant set $\rho\Phi - \rho'\Phi' = \zeta$.

The Hamiltonian formulation above has an equivalent Lagrangian formulation. Given suitable functions $\eta(x,\, t)$ and $\zeta(x,\, t)$ with fixed t-endpoints for η and convergent x-integrals, the set of equations (5) may be formally recovered by setting the first derivation of the functional

$$\mathcal{L} = \int_0^{\frac{2\pi}{\omega}} \Big[K - V\Big]\, \mathrm{d}t \tag{8}$$

equal to zero. Recalling finite-dimensional Hamiltonian systems, recovering (5) from (8) is analogous to Hamilton's Principle of Least Action.

Functional $\mathcal{L}$ can be computed through a weakly nonlinear analysis. Functions $\eta(x,t)$, $\phi(x,t)$, $\phi'(x,t)$ and $\zeta(x,t)$ can be expanded in double Fourier series in x and t. Details can be found for example in Dias and Bridges (1990). For the present analysis it is sufficient to retain two terms in the expansions. Considering wavelengths of $2\pi/k$ and time periods of $2\pi/\omega$, rescaling time by $t \mapsto \omega t$, and averaging over space and time, the correct reduced $\mathcal{L}$ to degree four in the amplitudes is given by

$$\mathcal{L}(E,M,\omega) = \frac{(\omega^2-\omega_0^2)\,\rho\,(T'+\varrho T)}{4kTT'}\,E - \frac{k^2\rho g\,(1-\varrho)}{4T^2T'^2(T'+\varrho T)}\,(\alpha E^2 + \beta M^2)\,, \tag{9}$$

where

$$\alpha(\varrho, k\delta, k\delta') = \tau(1)+\tau(2)+\tau(3)\,, \tag{10}$$
$$\beta(\varrho, k\delta, k\delta') = \tau(1)-\tau(2)+\tau(4)\,, \tag{11}$$

with

$$\begin{aligned}
\tau(1) &= \frac{[T'^2(3-T^2)-\varrho T^2(3-T'^2)]^2}{32\,TT'(T+\varrho T')}\,,\\
\tau(2) &= -\frac{[T'^2(1+T^2)-\varrho T^2(1+T'^2)]^2}{16\,(T'+\varrho T)}\,,\\
\tau(3) &= -\tfrac{1}{8}\,(T'^3+\varrho T^3)\,,\\
\tau(4) &= -\tfrac{1}{8}\,[T'^3(1-4T^2)+\varrho T^3(1-4T'^2)]\,.
\end{aligned}$$

Here, $T=\tanh k\delta$, $T'=\tanh k\delta'$ and ϱ is simply the density ratio ρ'/ρ. Quantity ω_0 denotes the linear frequency given by

$$\omega_0^2 = \frac{gkTT'\,(1-\varrho)}{T'+\varrho T}\,,$$

while quantities E and M are given by

$$E = |A|^2+|B|^2\,, \quad M = |B|^2-|A|^2\,, \tag{12}$$

where A and B are the first order complex Fourier series coefficients of

$$\eta = \mathrm{Re}\Big[A\,\mathrm{e}^{-\mathrm{i}(t-kx)} + B\,\mathrm{e}^{-\mathrm{i}(t+kx)} + \cdots\Big]\,. \tag{13}$$

Therefore, E and M may be thought of, respectively, as measures of the energy and momentum of the waves. The fact that amplitudes A and B appear only in combinations E and M comes from the symmetries of the problem, as was shown by Bridges and Dias (1990) who also predicted the general form of $\mathcal{L}$. Equation (9) is identical to equation (5.4) in Dias and Bridges (1994), in the limit as $k\delta$, $k\delta' \to \infty$ (setting their $b=0$).

3. Analysis of the normal form

Next, the so-called normal form of the problem, that is the system of equations relating the amplitudes of the waves to the angular wave frequency, is derived. Following definitions (12) and setting to zero the derivatives of $\mathcal{L}$ (9) with respect to $\overline{A}$, $\overline{B}$, A and B (with bar denoting complex conjugates) yields

$$A\Big[(\omega^2-\omega_0^2) - \frac{2k^3g\,(1-\varrho)}{TT'(T'+\varrho T)^2}\Big\{(\alpha+\beta)|A|^2+(\alpha-\beta)|B|^2\Big\}\Big] = 0\,, \tag{14}$$

$$B\Big[(\omega^2-\omega_0^2) - \frac{2k^3g\,(1-\varrho)}{TT'(T'+\varrho T)^2}\Big\{(\alpha-\beta)|A|^2+(\alpha+\beta)|B|^2\Big\}\Big] = 0\,, \tag{15}$$

and their complex conjugates. Looking at (13), there are three cases, as far as amplitudes A and B are concerned, to consider. Each case represents a different class of waves. One may see that $A=0$

corresponds to TWs with direction to the left (TW^-), whereas $B = 0$ corresponds to TWs with direction to the right (TW^+). Equations (12) show that TW^- are characterized by $E = M = |B|^2$, whereas TW^+ are characterized by $E = -M = |A|^2$. Case $|A| = |B|$ corresponds to SW, which may be viewed as a superposition of a TW^+ and a TW^- of equal amplitudes. These are characterized by $M = 0$. Finally, $A \neq B$ is a degenerate case, possible only if $\beta = 0$, and corresponds to degenerate mixed waves (MWs). This case has been given particular attention in Bridges and Dias (1990) for water waves. For all three classes, the normal form may be simplified. In particular, for TWs (for both TW^+ and TW^-) (12)–(13) yield the following nonlinear dispersion relation:

$$\omega^2 = \omega_0^2 \left\{1 + \frac{2k^2}{T^2T'^2(T' + \varrho T)}(\alpha + \beta)E\right\}, \quad (E = |A|^2 \text{ or } |B|^2). \tag{16}$$

It is worth mentioning that (16) is in agreement with (5.11) in Miles (1986) when $1-\varrho$ is *small* (Boussinesq limit), and with (2.1.4) ($k\delta$, $k\delta' \to \infty$), (2.1.6) ($k\delta = k\delta'$), (2.1.7) (either $k\delta$ or $k\delta' \to \infty$) and (2.1.9) ($k\delta = k\delta'$, Boussinesq limit) in Thorpe (1968*a*).

For SWs, (14) and (15) become identical and yield the following nonlinear dispersion relation:

$$\omega^2 = \omega_0^2 \left\{1 + \frac{2k^2}{T^2T'^2(T' + \varrho T)}\alpha E\right\}, \quad (E = 2|A|^2 = 2|B|^2). \tag{17}$$

Again, we mention the agreement of (17) with dispersion relations (2.1.2) (Boussinesq limit), (2.1.4) (either $k\delta$ or $k\delta' \to \infty$) and (1.4) ($k\delta$, $k\delta' \to \infty$) in Thorpe (1968*b*).

Finally, for MWs ($\beta = 0$), the corresponding nonlinear dispersion relation following from (14)–(15) is given by

$$\omega^2 = \omega_0^2 \left\{1 + \frac{2k^2}{T^2T'^2(T' + \varrho T)}\alpha E\right\}, \quad (E = |A|^2 + |B|^2). \tag{18}$$

This degenerate class of periodic waves is characterized by two dominant amplitudes and may be thought of as partially reflected waves which are the product of a superposition of a TW^+ and a TW^- of differing amplitudes. Clearly, TW^+s (resp. TW^-s) are a limiting case of MWs when $B = 0$ (resp. $A = 0$), while SWs correspond to MWs with $|A| = |B|$. Hence the latter provide a connection between branches of TW^+s and TW^-s and branches of SWs.

Dispersion relations (16) and (17) may yield branching (bifurcation) diagrams for TWs and SWs. These are frequency (or wavespeed) - amplitude (or energy) diagrams and the branching behavior depends on the sign of coefficients $\alpha + \beta$ for TWs and α for SWs. There exist values of parameters ϱ, $k\delta$ and $k\delta'$ through which these coefficients change sign. This very property renders $\alpha + \beta$ and α important towards the branching behavior of periodic waves.

Moreover, note that coefficients $\alpha + \beta$ and α appear in the amplitude equations which describe nonlinear modulations of the waves. Therefore more emphasis on their importance will be given when stability of interfacial waves is discussed in § 5.

It turns out that all three coefficients ($\alpha + \beta$, α and β) may vanish for appropriate choices of ϱ, T and T'. This behavior differs from its counterpart for gravity surface waves, where only α vanishes (for $T = 0.7849$), while $\alpha + \beta$ and β are always positive. It also differs from the corresponding behavior of gravity interfacial waves in infinite depth, where $\alpha + \beta$ and β are always positive, while α is always negative. The interesting new feature here is the presence of MWs, which so far have been associated with capillarity, in the absence of the latter. These waves may, in fact, exist for all $\varrho > 0$, given appropriate depths.

For TWs, $\alpha + \beta$ may vanish only for values of $T'/T < 1.2838$. For $T'/T > 1.2838$, $\alpha + \beta$ is always positive. For SWs, α may vanish only when $T'/T < 1.8354$. For $T'/T > 1.8354$, α is always positive. Degeneracy $\beta = 0$ (responsible for MWs) may be present for $T'/T < 1.1839$ only.

Figure 2 shows the (T,T')-plane for cases $\varrho = 0.1$, $\varrho = 0.5$ and $\varrho = 0.99$. Curves $\alpha = 0$, $\alpha + \beta = 0$ and $\beta = 0$ partition the plane into four regions. Numbers in each region correspond to a different branching

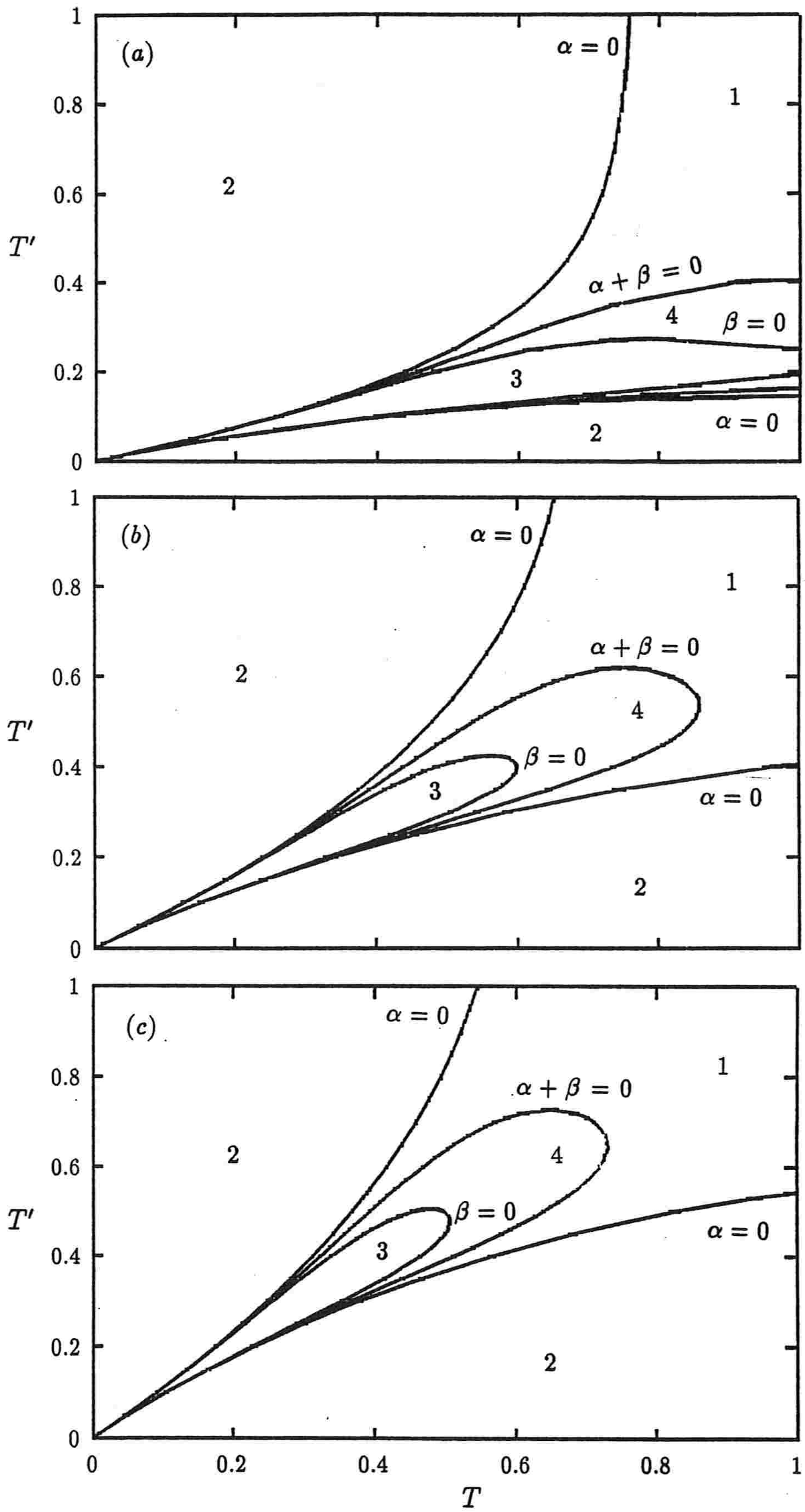

Figure 2. Signs of coefficients $\alpha+\beta$ and α in the parameter plane for different values of ϱ. (1) $\alpha+\beta>0$, $\alpha<0$, (2) $\alpha+\beta>0$, $\alpha>0$, (3) $\alpha+\beta<\alpha<0$, (4) $\alpha<\alpha+\beta<0$. (*a*) $\varrho=0.1$, (*b*) $\varrho=0.5$, (*c*) $\varrho=0.99$.

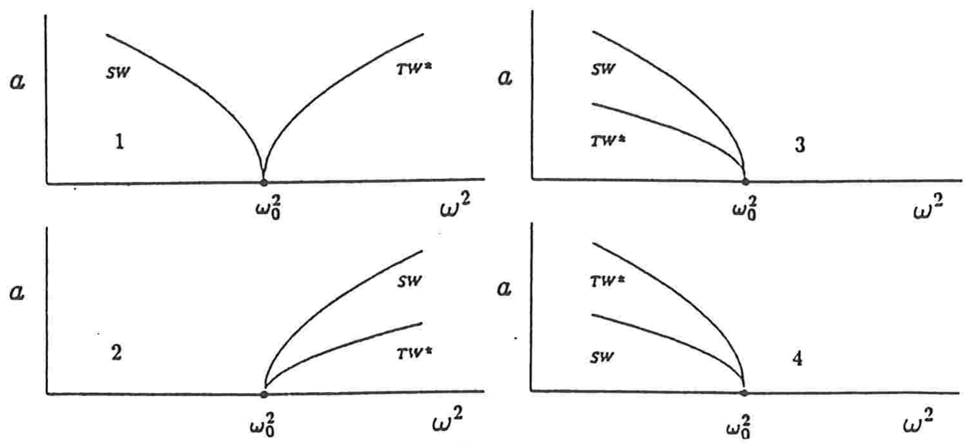

Figure 3. Corresponding branching behavior for $TW^{\pm}$ and SW for four regions shown in Figure 2 (a denotes amplitude).

behavior shown in Fig. 3. As demonstrated, for high values of ϱ (Fig. 2(c)), case which may be of interest to the oceanographer or the limnologist, MWs may only exist for a limited range of T and T' (i.e for a limited range of depths of the two fluids) while for low values of ϱ (Fig. 2(a)), they exist for all T. The same applies for TWs and SWs with branching behavior as in 3 and 4 of Fig. 3.

4. Multiple scales analysis

We now turn attention to modulational stability of interfacial waves. In order to study nonlinear interactions of small-amplitude waves, we follow the well-known method of multiple scales expansions. Note that y-dependence is considered hereafter. Spatial and temporal scales are introduced as follows:

$$X = \varepsilon x\,, \quad Y = \varepsilon y\,, \quad \mathcal{T} = \varepsilon t\,, \quad \tau = \varepsilon^2 t\,,$$

where ε is a small ordering parameter given (for example) by kA, with k and A being respectively the wavenumber and the wave amplitude. We assume that $\varepsilon << \min\{(k\delta)^2, (k\delta')^2\}$, where $k\delta$ and $k\delta'$ are $O(1)$.

We then consider asymptotic expansions of η, ϕ and ϕ' in ε as follows:

$$\eta(x,y,t;\varepsilon) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{l=-n}^{n} \varepsilon^n \eta_{[nml]} e^{i(mkx-l\omega t)}\,, \tag{19}$$

$$\phi(x,y,z,t;\varepsilon) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{l=-n}^{n} \varepsilon^n \phi_{[nml]} e^{i(mkx-l\omega t)}\,, \tag{20}$$

$$\phi'(x,y,z,t;\varepsilon) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{l=-n}^{n} \varepsilon^n \phi'_{[nml]} e^{i(mkx-l\omega t)}\,, \tag{21}$$

where $*_{[nml]} = *_{[nml]}(z, X, Y, \mathcal{T}, \tau) = \overline{*}_{[n(-m)(-l)]}$ (with bar denoting complex conjugate) and ω henceforth denotes linear frequency.

Substituting (19)–(21) into (1)–(4) and collecting coefficients of successive powers of ε (up to order 3 which suffices for our purpose), we obtain sets of perturbation equations. Use of these equations at orders ε, ε^2 and ε^3 provides expressions for $\phi_{[nml]}$ and $\phi'_{[nml]}$. For details see Christodoulides and Dias (1995).

The next step is to obtain expressions for coefficients $\eta_{[nml]}$ through the dynamic boundary condition

(4). Collecting coefficients of $e^{i(mkx-l\omega t)}$ (for all m, l), at each order of ε results in,

$$\begin{aligned} \eta_{[100]} &= 0\,, \\ \eta_{[11(\pm 1)]} &= \pm\frac{i\omega}{gr}(A^{\pm} - \varrho A'^{\pm})\,, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \eta_{[200]} &= -\frac{1}{gr}(\phi_{[100]\mathcal{T}} - \varrho\phi'_{[100]\mathcal{T}} - \frac{2\varrho k\omega^2}{(gr)^2}\Big[T(\overline{A^+}A'^+ + \overline{A^-}A'^-) - T'(A^+\overline{A'^+} + A^-\overline{A'^-})\Big] \\ &\quad -\frac{k^2}{gr}\Big(1 + T^2 - \frac{2T\omega^2}{grk}\Big)(|A^+|^2 + |A^-|^2) + \frac{\varrho k^2}{gr}\Big(1 + T'^2 - \frac{2\varrho T'\omega^2}{grk}\Big)(|A'^+|^2 + |A'^-|^2)\,, \\ \eta_{[21(\pm 1)]} &= \frac{1}{gr}\Big[\Big(\pm i\omega D^{\pm} - A^{\pm}_{\mathcal{T}}\Big) - \varrho\Big(\pm i\omega D'^{\pm} - A'^{\pm}_{\mathcal{T}}\Big)\Big]\,, \\ \eta_{[220]} &= \frac{k^2}{gr}\Big(1 - T^2 + \frac{2T\omega^2}{grk}\Big)A^+A^- - \frac{\varrho k^2}{gr}\Big(1 - T'^2 + \frac{2\varrho T'\omega^2}{grk}\Big)A'^+A'^- \\ &\quad -\frac{\varrho k(T - T')\omega^2}{(gr^2)}(A^+A'^- + A^-A'^+)\,, \\ \eta_{[202]} &= \frac{2i\omega}{gr}(J - \varrho J') + \frac{\varrho k(T - T')\omega^2}{(gr)^2}(A^+\overline{A'^-} + \overline{A^-}A'^+) \\ &\quad -\frac{k^2}{gr}\Big(1 + T^2 + \frac{2T\omega^2}{grk}\Big)A^+\overline{A^-} + \frac{\varrho k^2}{gr}\Big(1 + T'^2 + \frac{2\varrho T'\omega^2}{grk}\Big)A'^+\overline{A'^-}\,, \\ \eta_{[22(\pm 2)]} &= \pm\frac{2i\omega}{gr}(F^{\pm} - \varrho F'^{\pm}) + \frac{\varrho k(T - T')\omega^2}{(gr)^2}A^{\pm}A'^{\pm} \\ &\quad +\frac{k^2}{2gr}\Big(1 - T^2 - \frac{2T\omega^2}{grk}\Big)A^{\pm^2} - \frac{\varrho k^2}{2gr}\Big(1 - T'^2 - \frac{2\varrho T'\omega^2}{grk}\Big)A'^{\pm^2}\,, \end{aligned} \tag{23}$$

where $r = 1 - \varrho$.

Now, in order to determine the unknown amplitudes in (22)–(23), use is made of the kinematic conditions (3). At order ε we obtain

$$A'^{\pm} = -\frac{T}{T'}A^{\pm} \tag{24}$$

and the dispersion relation (recall § 3)

$$\omega^2 = \frac{grkTT'}{T' + \varrho T}\,. \tag{25}$$

Consequently the group velocity (useful in what follows) is given by

$$c_g \equiv \frac{d\omega}{dk} = \frac{gr[TT'(T' + \varrho T) + k\delta(1 - T^2)T'^2 + \varrho k\delta'(1 - T'^2)T^2]}{2\omega(T' + \varrho T)^2}\,. \tag{26}$$

Order ε^2 yields

$$A^{\pm}_{\chi^{\mp}} = 0\,, \tag{27}$$

where we have introduced variables $\chi^{\pm} \equiv X \mp c_g\mathcal{T}$, natural transformations for TWs, as mentioned in the introduction.

Finally, at order ε^3 are recovered evolution equations in $\phi_{[100]}$, $\phi'_{[100]}$ and $A^{\pm}$. In particular, for $m = l = 0$, we obtain

$$\begin{aligned} \phi_{[100]\mathcal{T}\mathcal{T}} - \varrho\phi'_{[100]\mathcal{T}\mathcal{T}} - \delta gr(\phi_{[100]XX} + \phi_{[100]YY}) &= a(|A^+|^2 - |A^-|^2)_X\,, \\ \phi_{[100]\mathcal{T}\mathcal{T}} - \varrho\phi'_{[100]\mathcal{T}\mathcal{T}} + \delta' gr(\phi'_{[100]XX} + \phi'_{[100]YY}) &= a'(|A^+|^2 - |A^-|^2)_X\,, \end{aligned} \tag{28}$$

where

$$\begin{aligned} a &= \frac{k^2}{T'^2}\Big\{c_g((1 - T^2)T'^2 - \varrho(1 - T'^2)T^2) + \frac{2grTT'^2}{\omega}\Big\}\,, \\ a' &= \frac{k^2}{T'^2}\Big\{c_g((1 - T^2)T'^2 - \varrho(1 - T'^2)T^2) - \frac{2grT^2T'}{\omega}\Big\}\,, \end{aligned}$$

and for $m = 1$, $l = \pm 1$,

$$\begin{aligned}
2\mathrm{i}dA^+_\tau &= -\frac{c_g d}{k}A^+_{YY} + eA^+_{\chi^+\chi^+} + \frac{2k\omega}{T'} < T'\phi_{[100]X} + \varrho T\phi'_{[100]X} >_{\chi^-} A^+ \\
&\quad + f < \phi_{[100]T} - \varrho\phi'_{[100]T} >_{\chi^-} A^+ + h|A^+|^2A^+ + j < |A^-|^2 >_{\chi^-} A^+ , \\
-2\mathrm{i}dA^-_\tau &= -\frac{c_g d}{k}A^-_{YY} + eA^-_{\chi^-\chi^-} - \frac{2k\omega}{T'} < T'\phi_{[100]X} + \varrho T\phi'_{[100]X} >_{\chi^+} A^- \\
&\quad + f < \phi_{[100]T} - \varrho\phi'_{[100]T} >_{\chi^+} A^- + h|A^-|^2A^- + j < |A^+|^2 >_{\chi^+} A^- ,
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
d &= \omega\frac{T' + \varrho T}{T'} , \\
e &= -d\frac{\partial c_g}{\partial k} , \\
f &= -\frac{\omega^2 k}{grTT'^2}((1 - T^2)T'^2 - \varrho(1 - T'^2)T^2) , \\
h &= -\frac{k^4((1 - T^2)T'^2 - \varrho(1 - T'^2)T^2)^2}{T'^3(T' + \varrho T)} \\
&\quad - \frac{k^4}{2TT'^4(T + \varrho T')}\Big\{T'^4(-9 + 10T^2 - 9T^4) + \varrho^2T^4(-9 + 10T'^2 - 9T'^4) + \varrho(18T^2T'^2 \\
&\qquad + 4TT'^5 - 6T^2T'^4 - 6T^4T'^2 + 4T^5T' - 8T^3T'^5 + 2T^4T'^4 - 8T^5T'^3)\Big\} , \\
j &= -\frac{k^4((1 - T^2)T'^2 - \varrho(1 - T'^2)T^2)^2}{T'^3(T' + \varrho T)} - \frac{k^4((1 + T^2)T'^2 - \varrho(1 + T'^2)T^2)^2}{T'^3(T' + \varrho T)} - 4k^4T^2 - 4\varrho k^4\frac{T^3}{T'} .
\end{aligned}$$

Note that the other amplitudes in (22)–(23) have been eliminated along the way. In particular, $D^\pm$ has been eliminated by demanding that expansions (19)–(21) remain asymptotic for x and t of $O(\varepsilon^{-2})$ which is the case if and only if the spatial averages (over a period) $< D^\pm_{\chi^\mp} >_{\chi^\mp}$ are zero.

Next, following the arguments of Pierce and Knobloch (1994), $\phi_{[100]}$ and $\phi'_{[100]}$ may be decomposed as follows:

$$\begin{aligned}
\phi_{[100]} &= \phi^+(\chi^+) + \phi^-(\chi^-) + \psi(\tau)\,T , \\
\phi'_{[100]} &= \phi'^+(\chi^+) + \phi'^-(\chi^-) + \psi'(\tau)\,T .
\end{aligned} \tag{30}$$

Note that it is not difficult to show that unknown functions ψ and ψ' are identically zero for TWs. They are, however, necessary for SWs which cannot avoid explicit dependence on time at leading order and are actually related to the mean level of the interface between the two fluids ($\eta_{[200]}$ in (23)). Substituting (30) into (28) and averaging over a period in Y, we obtain

$$\begin{aligned}
< \phi^\pm_{\chi^\pm} >_Y &= \pm\left\langle p|A^\pm|^2\right\rangle_Y , \\
< \phi'^\pm_{\chi^\pm} >_Y &= \pm\left\langle p'|A^\pm|^2\right\rangle_Y ,
\end{aligned} \tag{31}$$

where

$$p = \frac{(\varrho c_g^2 - \delta' gr)\,a - \varrho c_g^2\,a'}{\delta\delta'(gr)^2 - c_g^2(\delta\varrho + \delta')gr} , \quad p' = \frac{c_g^2\,a - (c_g^2 - \delta gr)\,a'}{\delta\delta'(gr)^2 - c_g^2(\delta\varrho + \delta')gr} ,$$

provided that $c_g^2 \neq \delta\delta' gr/(\delta\varrho + \delta')$ (which is actually the case for the waves under investigation).

To determine ψ and ψ', which are related to the mean fluid level, we require that the latter be zero (this assures that the total volume of the system remains unaltered - see Pierce and Knobloch 1994), that is $\left\langle < \eta_{[200]} >_X \right\rangle_Y = 0$. Then,

$$\psi - \varrho\psi' = \left\{\frac{k^2}{T'^2}(-(1 - T^2)T'^2 + \varrho(1 - T'^2)T^2) + c_g(p - \varrho p')\right\}\left\langle < |A^+|^2 >_{\chi^+} + < |A^-|^2 >_{\chi^-}\right\rangle_Y . \tag{32}$$

Finally, upon rescaling as follows

$$\tau^* = \frac{\tau}{2d} , \quad A^{*\pm}(\chi, Y, \tau^*) = A^\pm(\chi^\pm, Y, \tau) ,$$
$$\phi^{*\pm}(\chi, Y, \tau^*) = \phi^\pm(\chi^\pm, Y, \tau) , \quad \phi'^{*\pm}(\chi, Y, \tau^*) = \phi'^\pm(\chi^\pm, Y, \tau) ,$$

and dropping the stars, (29) and (28) yield the desired system of evolution equations:

$$\begin{aligned} (c_g^2 - \delta gr)\phi^{\pm}_{XX} - \varrho c_g^2 \phi'^{\pm}_{XX} - \delta gr \phi^{\pm}_{YY} &= \pm a|A^{\pm}|^2_X, \\ (\varrho c_g^2 - \delta' gr)\phi'^{\pm}_{XX} - c_g^2 \phi^{\pm}_{XX} - \delta' gr \phi'^{\pm}_{YY} &= \mp a'|A^{\pm}|^2_X, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \pm \mathrm{i} A^{\pm}_{\tau} = & -\frac{c_g d}{k} A^{\pm}_{YY} + eA^{\pm}_{XX} + h|A^{\pm}|^2 A^{\pm} + j < |A^{\mp}|^2 >_X A^{\pm} \\ & + f(\psi - \varrho\psi')A^{\pm} \pm \Big\{(2k\omega - c_g f)\phi^{\pm}_X + (\frac{2k\omega T}{T'} + c_g f)\varrho \phi'^{\pm}_X \Big\} A^{\pm} \\ & \pm \Big\{(2k\omega + c_g f) < \phi^{\mp}_X >_X + (\frac{2k\omega T}{T'} - c_g f)\varrho < \phi'^{\mp}_X >_X \Big\} A^{\pm}, \end{aligned} \tag{34}$$

where $\psi - \varrho\psi'$ (recall: present only for SWs) is given in (32).

System (33)–(34) reduces to equations (1) in Pierce and Knobloch (1994) when $\varrho = 0$ (setting their $\mu = 0$). When the Y-dependence is absent, (33)–(34) reduce to

$$\pm \mathrm{i} A^{\pm}_{\tau} = eA^{\pm}_{XX} + \mu|A^{\pm}|^2 A^{\pm} + \nu < |A^{\pm}|^2 >_X A^{\pm} + \xi < |A^{\mp}|^2 >_X A^{\pm}, \tag{35}$$

where

$$\begin{aligned} \mu &= h + \Big[(2k\omega - c_g f)p + (\frac{2k\omega T}{T'} + c_g f)\varrho p'\Big], \\ \nu &= f\Big[\frac{k^2}{T'^2}(-(1-T^2)T'^2 + \varrho(1-T'^2)T^2) + c_g(p - \varrho p')\Big], \\ \xi &= j + (2k\omega p + \frac{2k\omega T}{T'}\varrho p') + f\Big[\frac{k^2}{T'^2}(-(1-T^2)T'^2 + \varrho(1-T'^2)T^2)\Big]. \end{aligned}$$

For TWs ($A^- = 0$ or $A^+ = 0$), the averaged terms vanish and coefficient μ in (35) is in agreement with $\alpha + \beta$ in the nonlinear dispersion relation (16) (except for the group velocity term). In addition, (35) recovers (3.5) in Tanaka (1982). For SWs ($A^+ = A^-$), the coefficient of the cubic term in (35) becomes $\mu + \nu + \xi$ and is in agreement with α in (17).

5. Stability with respect to modulations

In this section a threefold study of the stability with respect to modulations is attempted. It is natural to consider modulations either in both the longitudinal (χ) and transverse (Y) directions, or in the longitudinal direction only, or in the transverse direction only.

(a) Stability Criteria

The analysis is based on stability criteria, derived separately for each case. The derivation is omitted here since it follows a pattern similar to the one in Pierce and Knobloch (1994) for water waves. The following conditions are obtained:

(*a*) Longitudinal modulations:

$$e\mu \leq 0. \tag{36}$$

This condition is valid for both TWs and SWs and is identical to the criterion (24) derived by Pierce and Knobloch (1994) for water waves. The same condition was derived for interfacial waves both by Nayfeh (1976) (for capillary–gravity TWs in infinite-depth) and by Grimshaw and Pullin (1985) (for gravity TWs). Violation of (36) for traveling water waves results in the well-known Benjamin–Feir instability.

(*b*) Longitudinal and transverse (oblique) modulations:

$$e < 0, \quad \mu > 0, \quad h > 0, \quad h^2 - j^2 > 0, \tag{37}$$

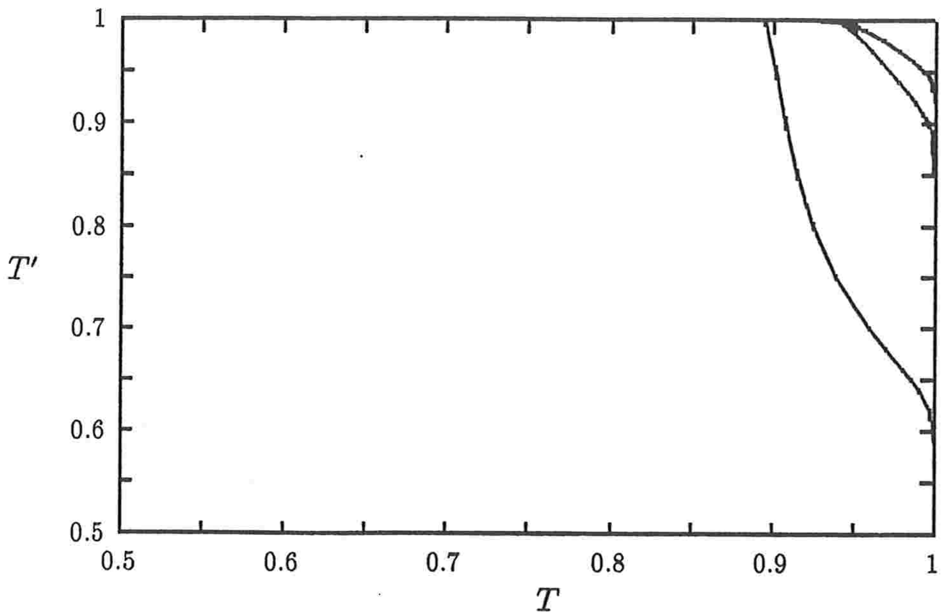

Figure 4. Stability diagram with respect to longitudinal modulations. Curve μ_1 is plotted for three different values of ϱ. From left to right: $\varrho = 0.1$, 0.5, 0.99. Both TWs and SWs are stable on the left of μ_1 and unstable on its right.

Again these are conditions valid for both TWs and SWs except for the last one which is only required for SWs.

(c) Transverse modulations:

$$h > 0\,, \quad h^2 - j^2 > 0\,. \tag{38}$$

These are conditions identical to the criterion (37) in Pierce and Knobloch (1994) for water waves. Condition $h^2 - j^2 > 0$ is only valid for SWs.

The stability criteria having been established, the effects of the two important physical parameters (second layer and finite depth) on the stability are studied, with particular attention given to the density ratio effects.

(b) Stability of gravity waves in finite depth

This case has been thoroughly studied by Tanaka (1982) and Grimshaw and Pullin (1985) for TWs, but to our knowledge there are no results for SWs.

The problem of interfacial gravity waves as described in § 2 is a 3-parameter problem, its parameters being ϱ, $k\delta$ and $k\delta'$. Therefore, fixing density ratio ϱ, stability diagrams may be obtained in the (T,T')-plane for all three cases described above. Such diagrams are obtained for three different values of ϱ namely 0.1, 0.5 and 0.99 (Boussinesq limit).

(a) The results for longitudinal modulations are presented in Fig. 4. Note that $e > 0$ for all ϱ. Hence, stability is established when $\mu < 0$. For each value of ϱ there exists exactly one critical curve μ_1 through which μ changes sign. This curve acts as barrier and all waves (meaning both TWs and SWs) on its left are stable while all waves on its right are unstable. In Fig. 4, curve μ_1 is plotted for the three values of ϱ mentioned above. Note that for a better visualization, not the whole (T,T')-plane is shown. It is clear that the larger the ϱ the larger the stable area in the (T,T')-plane, demonstrating the stabilizing role of ϱ. On the other hand, it is illustrated that letting both T and T' tend to 1 (i.e. the two depths to infinity) has a destabilizing effect. Finally, note that for the special case $\varrho = 0$ (water waves - not shown) the critical value is $k\delta = 1.363$ in agreement with Whitham's (1967) value.

(b) As $e > 0$ regardless of ϱ, the first condition of (37) is violated and hence all waves are unstable with

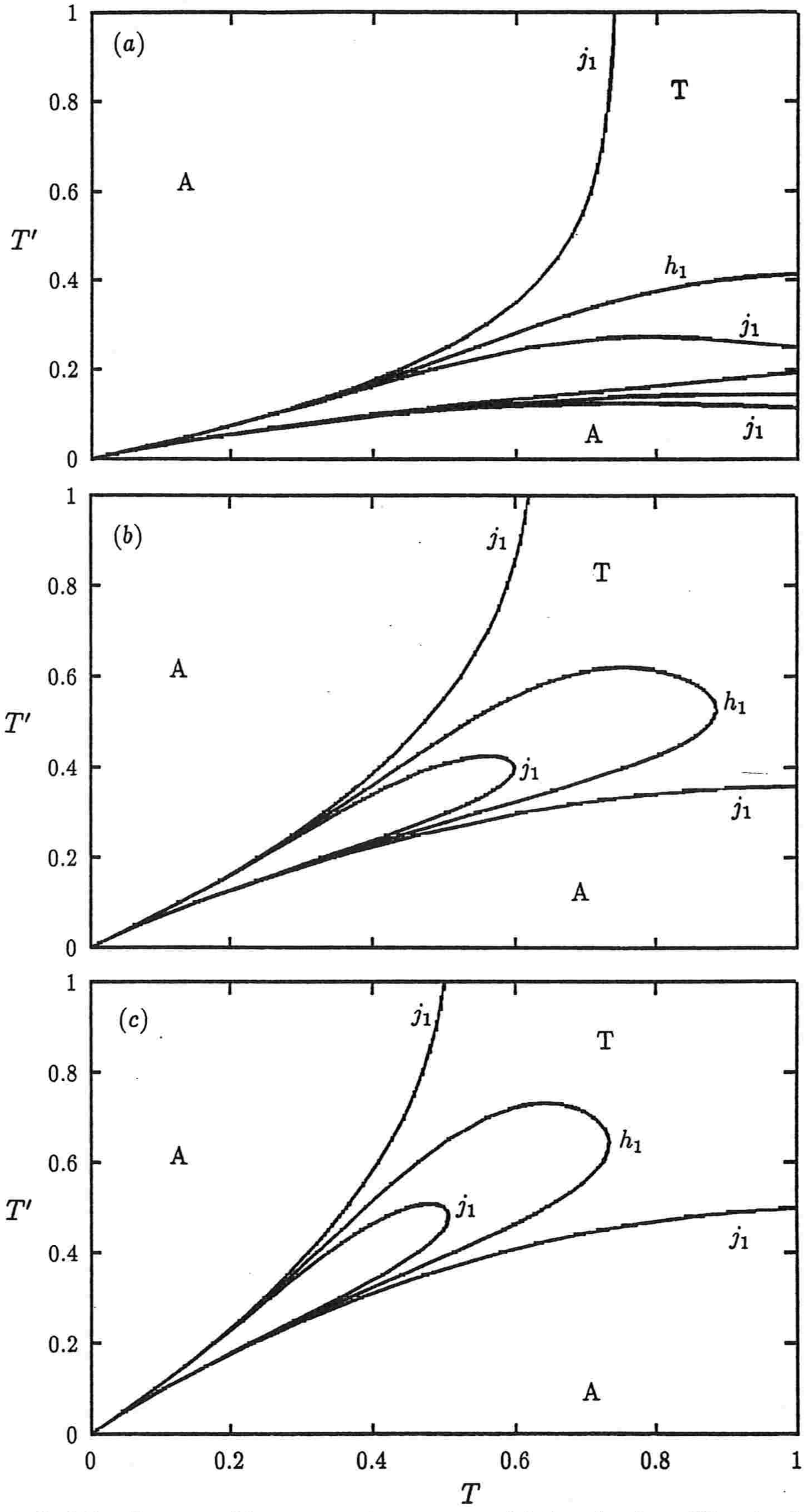

Figure 5. Stability diagrams with respect to transverse modulations for three different values of ϱ. (*a*) $\varrho = 0.1$, (*b*) $\varrho = 0.5$, (*c*) $\varrho = 0.99$. (A: stability regions for both *TW*s and *SW*s, T: stability regions for *TW*s only).

respect to oblique modulations. This verifies Grimshaw and Pullin's (1985) results for gravity TWs.

(c) When only transverse modulations are considered, the results are shown in Fig. 5. For $\varrho = 0.1$, there are two critical curves: h_1 consisting of two legs and corresponding to the vanishing of coefficient h, and j_1 consisting of four legs (two "insideänd two "outside" h_1) corresponding to the vanishing of coefficient $h^2 - j^2$. Notice that the results are (naturally) reminiscent of Fig. 2. Traveling waves are stable everywhere outside h_1, while SWs are stable in a smaller region of the plane as illustrated in Fig. 5(a). In the limiting case where both fluids are deep ($T = T' = 1$) TW are stable whereas SWs are unstable. For $T = 1$ ($k\delta \to \infty$), all waves are stable for $T' < 0.114$, whereas for $T' = 1$ ($k\delta' \to \infty$), all waves are stable for $T < 0.7415$. For $\varrho = 0.5$ (Fig. 5(b)), h_1 has become a loop as have the two legs of j_1 inside it. The stability pattern remains the same. The two extreme values in the (T,T')-plane are (1, 0.3584) and (0.6203, 1). Increasing the density ratio to $\varrho = 0.99$ (Fig. 5(c)), results in the two loops having decreased in size. Consequently TWs are stable in a larger region than before. Moreover, the two extreme points in the (T,T')-plane have moved to (1, 0.498) and (0.502, 1). Comparing this case to case (a), one concludes that interfacial waves in oceans or lakes are more stable than interfacial waves of small density ratio (which behave like surface waves) when the lower fluid is much deeper than the upper fluid, but less stable when the upper fluid is deeper than the lower fluid.

Finally, comparing the stability regions in Fig. 5 to the regions in Fig. 2, one may draw the conclusion that all stable periodic (traveling or standing) waves have the physical property of increasing amplitudes with increasing angular frequency (Fig. 3 (2)).

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