Gap-solitons in a three-layered stratified flow

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Abstract

In this paper, we give an explicit asymptotic construction of a class of solitary waves, widely known as gap-solitons in other physical contexts, for a certain three-layered fluid flow. The essential ingredients are the existence of a spectral gap between two branches of the dispersion relation, and the development of a set of envelope equations to describe weakly nonlinear waves, whose carrier frequency and wavenumber belong to the centre of this gap. Here we describe the gap-soliton solutions to this set of envelope equations. For the special case of particular interest when the envelope and carrier speeds are identical, so that the gap-soliton is a steady travelling wave of the full fluid system, we show that there is large class of such gap-solitons.

Keywords:

1 Introduction

It is well-known that in fluid flows, there are two main classes of solitary waves. Each of these can be regarded as a bifurcation from those points in the linear spectrum where the group velocity and phase velocity are equal (see, for instance, [1, 2, 3]). When this bifurcation occurs in the long-wave limit, that is at the zero wavenumber limit, it typically leads to the Korteweg-de

Vries equation, and its well-known solitary wave solutions. The alternative situation is when the bifurcation occurs at a finite non-zero wavenumber. In this case, the outcome for weakly nonlinear waves is related to the nonlinear Schrödinger (NLS) equation, and leads to envelope solitary waves; these waves have an underlying rapidly varying oscillation, contained within a smoothly varying envelope; consequently their decaying tails are also oscillatory. In both these scenarios, the bifurcation occurs into the "gap" in the wavenumber space, that is for phase velocities for which the linear dispersion relation does not have any real wavenumber solutions.

In this paper, we are concerned with the second case, but for that special situation when there are two branches of the dispersion relation in close proximity, so that the gap is narrow in a sense which will be defined precisely below. Such situations are well-known in nonlinear optics, and lead to the so-called *gap-soliton* solutions (see, for instance, [4]). However, apart from an initial study by Grimshaw and Malomed [5] of a certain model system consisting of two coupled Korteweg-de Vries equations, there presence in fluid flows has yet to be determined. In Grimshaw and Christodoulides [6] (hereafter GC) we identified the circumstances when a spectral gap can arise in a three-layered fluid flow, and developed a set of envelope equations to describe weakly nonlinear waves associated with such a gap. Here we extend that analysis to show explicitly that these envelope equations allow for a large class of solitary wave solutions, which we shall call gap-solitons.

To set the scene, let us suppose that the spectral gap is centred at a wavenumber k_0 and corresponding frequency ω_0. Then in the vicinity of k_0, ω_0 the linear dispersion relation takes the generic form

$$(\omega - \omega_0 - V_1(k - k_0)(\omega - \omega_0 - V_2(k - k_0)) = \gamma^2 \,, \tag{1}$$

where γ is a small parameter describing the width of the spectral gap. Here $V_{1,2}$ are the group velocities at $k = k_0$ of each branch in the limit $\gamma = 0$, and it is understood that $V_1 \neq V_2$. The limit when $V_1 - V_2 = O(\gamma)$ requires a different treatment from that we consider here, and typically leads to a set of coupled NLS equations (see, for instance, [7] and the references therein). The situation is sketched in Figure 1 for the case when $V_1 = -V_2$.

In the limit when $\gamma = 0$, the dispersion relation (1) collapses to two intersecting straight lines,

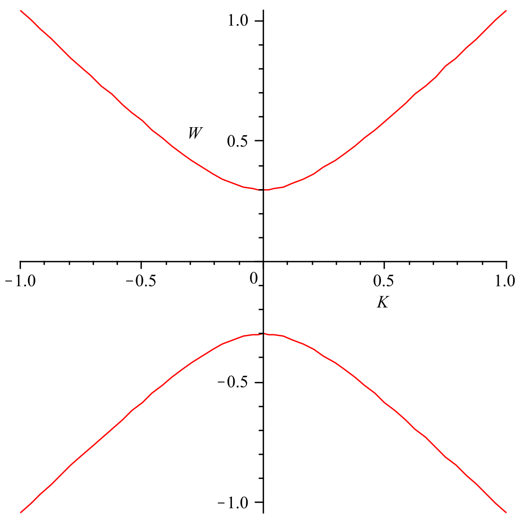

Figure 1: A typical spectral gap, where $W = \omega - \omega_0, K = k - k_0$.

corresponding to two uncoupled wave modes. There are two cases to consider, depending on whether the system supports one or two wave modes at the resonance point $\omega = \omega_0, k = k_0$ (see GC). Our concern here is with the latter case, where the wave modes can be represented by

$$A(x,t) \exp(ik_0 x - i\omega_0 t)\,, \;\; B(x,t) \exp(ik_0 - i\omega_0 t)\,. \tag{2}$$

Here $A(x,t), B(x,t)$ are the slowly-varying envelopes, and as shown in GC are expected generically to satisfy a coupled equation set of the form

$$i(A_t + V_1 A_x) + \gamma B + \mathrm{N}_1 \;\; = \;\; 0\,, \tag{3}$$

$$i(B_t + V_2 B_x) + \gamma A + \mathrm{N}_2 \;\; = \;\; 0\,. \tag{4}$$

Here the form of the linear terms is dictated by the linear dispersion relation (1), while N_1, N_2 represent cubic nonlinear terms of the general form $|A|^2 A, |B|^2 A, B^2 A^*$, etc where * denotes the complex conjugate, together with a set of terms of the general form MA, etc, where each M represent a mean-flow variable, satisfying an equation of the form

$$M_t + V_0 M_x = |A|^2_x\,, \tag{5}$$

where V_0 is a long-wave phase speed.

For any specific fluid flow system, the aim then is to determine the presence of the linear resonance conditions where there are two independent wave modes at the resonance point, and then to determine the specific forms of N_1, N_2. This program was carried out by GC for a three-layered fluid flow. Here we continue that analysis by showing explicitly that the resulting envelope system does indeed support a large family of gap-solitons. In section 2 we give a brief summary of the envelope equations. Then in section 3 we obtain the explicit gap-soliton solutions. Particular interest attaches the case when the speed of the gap-soliton is exactly equal to the phase speed ω_0/k_0 of the carrier wave, as in this case we have established the existence, at least asymptotically, of a steady traveling solitary wave solution of the full system. This issue is taken up in section 5, where we find a large class of such solutions. We conclude in section 6.

2 Formulation and weakly nonlinear analysis

For convenience, we present a summary of the analysis in GC of a three-layered fluid flow. The fluid is composed of three layers, each of constant density ρ_i and with a basic constant horizontal velocity U_i $(i = 1, 2, 3)$, as shown in Figure 2. The two interfaces are described by $z = \eta(x, t)$ and $z = H_2 + \zeta(x, t)$. We assume that the fluid is inviscid and incompressible and that the flow in each layer is irrotational and two-dimensional, with velocity potentials ϕ_i $(i = 1, 2, 3)$. The governing equations are then given by

$$\phi_{ixx} + \phi_{izz} = 0\,, \quad (i = 1, 2, 3) \tag{6}$$

subject to the boundary conditions

$$\phi_{1z} = 0\,, \qquad (z = -H_1) \tag{7}$$

$$\phi_{3z} = 0\,, \qquad (z = H_2 + H_3) \tag{8}$$

where $w_i = \phi_{iz}$ are the vertical perturbation velocities in each layer.

At the two interfaces the kinematic conditions are given respectively by

$$\eta_t + (U_i + u_i)\eta_x = w_i\,, \qquad (i = 1, 2) \qquad \text{at} \quad z = \eta\,, \tag{9}$$

$$\zeta_t + (U_i + u_i)\zeta_x = w_i\,, \qquad (i = 2, 3) \qquad \text{at} \quad z = H_2 + \zeta\,, \tag{10}$$

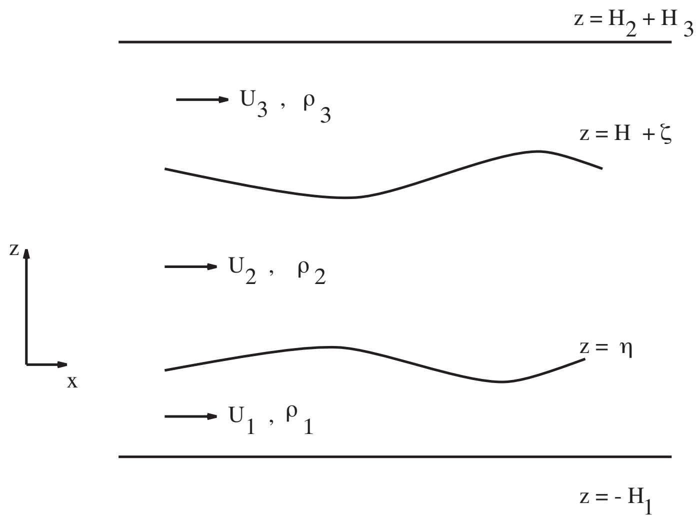

Figure 2: A sketch of the coordinate system for a three-layered fluid.

while the dynamic conditions are given respectively by

$$\rho_1(\phi_{1t} + U_1\phi_{1x} + \frac{1}{2}|\nabla\phi_1|^2 + g\eta) - \rho_2(\phi_{2t} + U_2\phi_{2x} + \frac{1}{2}|\nabla\phi_2|^2 + g\eta) = \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{3/2}}, \quad (11)$$

$$\rho_2(\phi_{2t} + U_2\phi_{2x} + \frac{1}{2}|\nabla\phi_2|^2 + g\zeta) - \rho_3(\phi_{3t} + U_3\phi_{3x} + \frac{1}{2}|\nabla\phi_3|^2 + g\zeta) = \frac{\sigma''\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}, \quad (12)$$

where $u_i = \phi_{ix}$ are the horizontal perturbation velocities in each layer and σ, σ' are the surface tension coefficients at each interface.

In GC we performed a weakly nonlinear analysis that studied solutions to the system (6)−(12) above of the form

$$\eta = \alpha A(X,T)e^{i\theta} + \alpha^2 A_2(X,T)e^{2i\theta} + c.c. + \alpha^2\bar{\eta}(X,T) + O(\alpha^3), \quad (13)$$

$$\zeta = \alpha B(X,T)e^{i\theta} + \alpha^2 B_2(X,T)e^{2i\theta} + c.c. + \alpha^2\bar{\zeta}(X,T) + O(\alpha^3), \quad (14)$$

where $\theta = kx - \omega t$, *c.c.* denotes the complex conjugate and the slow variables are defined by $X = \alpha^2 x$ and $T = \alpha^2 t$.

At the leading order the linearized system is obtained

$$D_1 A + EB = 0, \qquad EA + D_3 B = 0, \quad (15)$$

where $D_1(\omega,k), D_3(\omega.k), E(\omega,k)$ are defined in the Appendix by (56, 57, 58). This has the required form for a resonance between two wave modes, as discussed in the Introduction. We are concerned here with the case when there are two distinct modes at the resonance point, and so we require that $D_1 = D_2 = E = 0$, which is achieved by the requirements

$$\omega = kU_2 \,, \tag{16}$$

$$g(\rho_1 - \rho_2) + \sigma k^2 = \frac{\rho_1 k (U_2 - U_1)^2}{T_1} \,, \tag{17}$$

$$g(\rho_2 - \rho_3) + \sigma' k^2 = \frac{\rho_3 k (U_2 - U_3)^2}{T_3} \,. \tag{18}$$

The solution of this equation set determines the resonant frequency ω_0 and resonant wavenumber k_0. Hereafter we shall omit the "0" subscript for simplicity. Without loss of generality, we may set $U_2 = 0$ and choose $U_3 > 0$. Then the resonance condition (16) becomes $\omega = 0$, and the remaining conditions reduce to

$$U_1 = \mp\{\frac{T_1}{k}\frac{g(\rho_1 - \rho_2)}{\rho_1} + \frac{\sigma k^2}{\rho_1}\}^{1/2} \,, \tag{19}$$

$$U_3 = \{\frac{T_3}{k}\frac{g(\rho_2 - \rho_3)}{\rho_3} + \frac{\sigma' k^2}{\rho_3}\}^{1/2} \,. \tag{20}$$

The choice of sign in (19) corresponds to a shear flow ($U_1 < 0$) or a jet flow ($U_1 > 0$) respectively. For a given wavenumber k, the conditions (19, 20) define the basic velocities for which a resonance can occur. Alternatively, elimination of k defines a functional relationship between U_1, U_3 for which a resonance of this type can occur.

GC showed that the weakly nonlinear unfolding of this resonance leads to a coupled system of evolution equations, given by

$$iD_{1\omega}(A_T + V_1 A_X) + \gamma B + \mu_1 |A|^2 A + \nu_1 A\bar{\eta} - kD_{1\omega}A\bar{u}_1 = 0 \,, \tag{21}$$

$$iD_{3\omega}(B_T + V_3 B_X) + \gamma A + \mu_3 |B|^2 B + \nu_3 B\bar{\zeta} - kD_{3\omega}B\bar{u}_3 = 0 \,, \tag{22}$$

Here $D_{1\omega}, D_{3\omega}$ denote derivatives with respect to ω, evaluated at the resonant wavenumber (see 59), while V_1, V_3 are the group velocities on each interface at the resonant wavenumber, and are given by (60, 61) in the Appendix. The parameter $\gamma > 0$ is a tuning parameter for the resonance, formally defined by $E = \alpha^2\gamma$, so that $\omega = \alpha k\Omega$, where $\gamma = \rho_2 k\Omega^2/S_2$. The remaining coefficients, and the equations for the mean flow variables are given in the Appendix. Our concern here is to obtain steady travelling wave solutions of this equation set, analogous to the gap-soliton solutions found by Grimshaw and Malomed (1994) in a model problem.

3 Solitary waves

We seek travelling wave solutions of (21, 22) of the form

$$
\begin{aligned}
A(X,T) &= R(\xi)e^{i\phi(\xi)}e^{-i\Omega T}\,, &(23)\\
B(X,T) &= S(\xi)e^{i\psi(\xi)}e^{-i\Omega T}, &(24)\\
\bar{m}(X,\,T) &= \bar{m}(\xi), &(25)
\end{aligned}
$$

where $\xi = X - VT$, R, S are real-valued amplitudes and ϕ, ψ are real-valued phases, while $\bar{m}$ denotes the mean-flow variables presented in the Appendix. We shall assume further that $R, S \to 0$ as $\xi \to \pm\infty$, so that these will be solitary waves, that is gap-solitons.

Substitution of (23, 24, 25) into the mean flow equations (62)−(67) yields the following system of ordinary differential equations.

$$
\begin{aligned}
(U_1 - V)\bar{\eta}' + H_1\bar{u}'_1 &= \frac{2kU_1}{T_1}(|A|^2)'\,, &(26)\\
-V\bar{\eta}' - C &= 0\,, &(27)\\
-V\bar{\zeta}' + H_2\bar{u}'_2 - C &= 0\,, &(28)\\
(U_3 - V)\bar{\zeta}' + H_3\bar{u}'_3 &= -\frac{2kU_3}{T_3}(|B|^2)'\,, &(29)\\
\rho_1(U_1 - V)\bar{u}'_1 + \rho_2 V\bar{u}'_2 + g(\rho_1 - \rho_2)\bar{\eta}' &= -\rho_1 U_1^2 k^2 \frac{1 - T_1^2}{T_1^2}(|A|^2)'\,, &(30)\\
-\rho_2 V\bar{u}'_2 - \rho_3(U_3 - V)\bar{u}'_3 + g(\rho_2 - \rho_3)\bar{\zeta}' &= \rho_3 U_3^2 k^2 \frac{1 - T_3^2}{T_3^2}(|B|^2)'\,, &(31)
\end{aligned}
$$

where $'$ denotes differentiation with respect to ξ. Using the solitary-wave boundary condition, the solution of this system is

$$
\begin{aligned}
\bar{\eta} &= \eta_A|A|^2 + \eta_B|B|^2\,, &(32)\\
\bar{\zeta} &= \zeta_A|A|^2 + \zeta_B|B|^2\,, &(33)\\
\bar{u}_1 &= u_{1A}|A|^2 + u_{1B}|B|^2, &(34)\\
\bar{u}_3 &= u_{3A}|A|^2 + u_{3B}|B|^2, &(35)
\end{aligned}
$$

where the coefficients are given in the Appendix. Then substitution of (23, 24, 25) into (21, 22) yields

$$
iD_{1\omega}(V_1 - V)R' - D_{1\omega}(V_1 - V)\phi' R + (\mu_1 + \nu_1\eta_A - kD_{1\omega}u_{1A})|R|^2 R
$$

$$+(-\nu_1\eta_B - kD_{1\omega}u_{1B})|S|^2R + \Omega D_{1\omega}R = -\gamma S e^{i(\psi-\phi)}\,, \tag{36}$$

$$\begin{aligned} iD_{3\omega}(V_3 - V)S' - D_{3\omega}(V_3 - V)\psi' S + (\mu_3 + \nu_3\zeta_B - kD_{3\omega}u_{3B})|S|^2S \\ (-\nu_3\zeta_A - kD_{3\omega}u_{3A})|R|^2S + \Omega D_{3\omega}S = -\gamma R e^{i(\phi-\psi)}\,. \end{aligned} \tag{37}$$

First, the imaginary parts of (36, 37) give

$$D_{1\omega}(V_1 - V)R' = \gamma S \sin\Phi\,, \tag{38}$$

$$D_{3\omega}(V_3 - V)S' = -\gamma R \sin\Phi\,. \tag{39}$$

where $\Phi = \phi - \psi$. These can be integrated, assuming that both $R, S \to 0$ as $\xi \pm \infty$, to show that

$$D_{1\omega}(V_1 - V)R^2 = -D_{3\omega}(V_3 - V)S^2\,. \tag{40}$$

It follows that for a solution to exist we must have

$$D_{1\omega}(V_1 - V)D_{3\omega}(V - V_3) > 0\,. \tag{41}$$

Next we see that (38, 39) reduce to

$$(R^2)' = \tilde{\gamma} R^2 \sin\Phi\,. \tag{42}$$

where

$$\tilde{\gamma} = \pm\frac{2\gamma}{\sqrt{D_{1\omega}(V_1 - V)D_{3\omega}(V - V_3)}}\,, \tag{43}$$

and here $\pm$ corresponds to sign$\{D_{1\omega}(V_1 - V)\}$, where without loss of generality, we have assumed that R, S have the same sign (otherwise the transformation $\Phi \to \Phi + \pi$ restores the same analysis which follows below).

Second, the real parts of (36, 37) give

$$\begin{aligned} -D_{1\omega}(V_1 - V)\phi' + (\mu_1 + \nu_1\eta_A - kD_{1\omega}u_{1A})R^2 + (-\nu_1\eta_B - kD_{1\omega}u_{1B})S^2 \\ +\Omega D_{1\omega} = -\gamma\frac{S}{R}\cos\Phi\,, \end{aligned} \tag{44}$$

$$\begin{aligned} -D_{3\omega}(V_3 - V)\psi' + (\mu_3 + \nu_3\zeta_B - kD_{3\omega}u_{3B})S^2 + (-\nu_3\zeta_A - kD_{3\omega}u_{3A})R^2 \\ +\Omega D_{3\omega} = -\gamma\frac{R}{S}\cos\Phi\,, \end{aligned} \tag{45}$$

Then, using (40) we get

$$\Phi' = WR^2 + \tilde{\gamma}\cos\Phi - \tilde{\Omega}\,, \tag{46}$$

where

$$W = \frac{(\nu_1\eta_B + kD_{1\omega}u_{1B} + \nu_3\zeta_A + kD_{3\omega}u_{3A})}{D_{3\omega}(V_3 - V)}$$
$$+\frac{(\mu_1 + \nu_1\eta_A - kD_{1\omega}u_{1A})}{D_{1\omega}(V_1 - V)} + \frac{D_{1\omega}(V_1 - V)}{D_{3\omega}^2(V_3 - V)^2}(\mu_3 + \nu_3\zeta_B - kD_{3\omega}u_{3B}) \tag{47}$$

and

$$\tilde{\Omega} = \Omega\frac{V_3 - V_1}{(V_1 - V)(V - V_3)}\,.$$

Note that W changes sign with $\text{sign}\{D_{1\omega}(V_1 - V)\}$ and so it will be useful to write $\tilde{W} = \pm W$ accordingly.

Finally, we can eliminate R to obtain a single second order equation for Φ, whose first integral is

$$(\Phi')^2 = \frac{1}{2}\tilde{\gamma}^2\cos 2\Phi - 2\tilde{\gamma}\tilde{\Omega}\cos\Phi + \text{C.I.} \tag{48}$$

The constant of integration C.I. is found by letting $\xi \to \pm\infty$ in (46) to get

$$\text{C.I.} = \frac{1}{2}\tilde{\gamma}^2 + \tilde{\Omega}^2\,.$$

Equations (46, 48) can then be combined to show that

$$\Phi' = -\tilde{\gamma}\cos\Phi + \tilde{\Omega}\,, \tag{49}$$

and

$$WR^2 = 2\Phi'\,, \tag{50}$$

Integration of (49) then yields

$$\tan\frac{\Phi}{2} = \alpha[\tanh K\xi]^{\pm 1}\,, \tag{51}$$

where

$$\alpha^2 = \frac{\tilde{\gamma} - \tilde{\Omega}}{\tilde{\gamma} + \tilde{\Omega}} \quad \text{and} \quad K = -\frac{1}{2}\alpha(\tilde{\gamma} + \tilde{\Omega})\,.$$

Note that we must have $\tilde{\gamma}^2 > \tilde{\Omega}^2$ and the "$\pm$" case in (51) holds for $W(\tilde{\gamma} - \tilde{\Omega}) < (>) \ 0$ respectively, that is, as $\tilde{W} < (>) \ 0$. Finally, from (50) we get the solitary-wave solution

$$WR^2 = -(\tilde{\gamma} - \tilde{\Omega}) \frac{2\text{sech}^2(K\xi)}{1 + \alpha^2 \tanh^2(K\xi)}, \quad \text{for} \quad \tilde{W} < 0, \tag{52}$$

$$WR^2 = (\tilde{\gamma} - \tilde{\Omega}) \frac{2\text{sech}^2(K\xi)}{\tanh^2(K\xi) + \alpha^2}, \quad \text{for} \quad \tilde{W} > 0. \tag{53}$$

In each case there are two solutions for R, identical except for the sign, so that one may be interpreted as an elevation wave (say $R > 0$) and the other is then a depression wave ($R < 0$). With R known we can use (40) to determine S^2, and finally S, where we recall that the sign of S is required to be the same as that of R. But we now see from (51) that in the "+"-case, $\tilde{W} < 0$, $\Phi = 0$ at the wave crest where $\xi = 0$, and so the two interfaces are in phase. In the opposite "$-$"-case, $\tilde{W} > 0$, $\Phi = \pm\pi$ at the wave crest, and the two interfaces are now out of phase.

In general, this solution contains three free parameters, V, Ω, γ and although these are constrained by (41) for V and by the requirement that $\tilde{\gamma}^2 > \tilde{\Omega}^2$, there still remains a very large parameter space, in addition to the system parameters (the basic fluid densities, flow velocities and layer depths). Hence to make some progress, we shall consider a reduced class of solutions in the next section.

4 Steady solutions ($V = 0, \ \ \Omega = 0$)

Although the solitary-wave envelope is steady, the complete solution (13, 14) is not steady unless we also put $V = 0$, $\Omega = 0$, noting that the carrier wave has a phase velocity ω/k and here the resonance condition (16) with $U_2 = 0$ requires us to set $\omega = 0$. With $V = 0$, the condition (41) reduces to (see (59))

$$U_1 V_1 U_3 V_3 < 0, \tag{54}$$

or, on substituting for the group velocities $V_{1,3}$ from (60, 61),

$$(\Delta_1 - \frac{2\sigma k T_1}{\rho_1 U_1^2})(\Delta_3 - \frac{2\sigma' k T_3}{\rho_3 U_3^2}) < 0\,, \tag{55}$$

where

$$\Delta_1 = 1 - \frac{k H_1 T_1}{S_1^2}\,, \quad \Delta_3 = 1 - \frac{k H_3 T_3}{S_3^2}\,.$$

Note that in the absence of surface tension (that is $\sigma = \sigma' = 0$) the condition (55) cannot be satisfied for any system parameters. But the condition can be satisfied if σ is sufficiently large (small) and σ' is sufficiently small (large). Note that for $\Omega = 0$, $\alpha = 1$, $K = -\tilde{\gamma}/2$ and the solitary waves exist for all $\gamma > 0$.

We next present two representative cases, one for $\sigma' = 0$ and one for nonzero surface tension coefficients. Here we use dimensionless variables, denoted by a "hat" overbar, based on a length scale H_2 and a velocity scale $\sqrt{gH_2}$. The following notation is introduced to describe dimensionless surface tension,

$$\hat{\sigma} = \frac{\sigma k^2}{g(\rho_1 - \rho_2)}\,, \quad \hat{\sigma}' = \frac{\sigma' k^2}{g(\rho_2 - \rho_3)}\,.$$

In both cases we set $k = 1\,m^{-1}$, $H_2 = 50\,m$, $H_1 = H_3 = 500\,m$ and $\rho_{1,2,3} = 1010, 1000, 990\,kg\,m^{-3}$ respectively. In the first case $\hat{\sigma}' = 0$ yields $\hat{U}_3 = 0.014$ and $\hat{V}_3 = 0.007$. Then the effect of varying $\hat{\sigma}$ is shown in Table 1. The "$-$"("$+$")-sign for $\hat{U}_1$ corresponds to a shear(jet)-flow case. It is clear that steady solitons exist for values of $\hat{\sigma} > 1$ and fall in the "+"-case in (51) for moderate magnitudes of surface tension. For large values of $\hat{\sigma}$ (> 3.31) one observes "$-$"-case solitons.

Next, in the second case we set $\hat{\sigma}' = 0.40$ so that $\hat{U}_3 = 0.017$ and $\hat{V}_3 = 0.004$. The effect of varying $\hat{\sigma}$ is shown in Table 2. Again, the "$-$"("$+$")-sign for $\hat{U}_1$ corresponds to a shear(jet)-flow case. Steady solitons exist for values of $\hat{\sigma} > 1$ and fall into the "+"-case for $\hat{\sigma} < 1.38$. Then for larger magnitudes of surface tension, one observes "$-$"-case solitons.

$\hat{\sigma}$	$\hat{U}_1$	$\hat{V}_1$	sign($\tilde{W}$)	valid
0	∓ 0.0141	∓ 0.0070	$-$	N
0.20	∓ 0.0154	∓ 0.0051	$-$	N
0.40	∓ 0.0167	∓ 0.0036	$-$	N
0.51	± 0.0173	∓ 0.0028	$+$	N
0.99	∓ 0.0199	∓ 0.00005	$+$	N
1.01	∓ 0.0200	± 0.00005	$-$	Y
1.50	∓ 0.0222	± 0.0022	$-$	Y
2.00	∓ 0.0244	± 0.0041	$-$	Y
2.50	∓ 0.0263	± 0.0056	$-$	Y
3.30	∓ 0.0292	± 0.0078	$-$	Y
3.31	∓ 0.0292	± 0.0078	$+$	Y

Table 1: Region of valid solutions for $\hat{\sigma}' = 0$ when $k = 1\,m^{-1}$, $H_2 = 50\,m$, $H_1 = H_3 = 500\,m$, based on the criterion (55). The symbol "Y" means valid, and the symbol "N" means not valid.

Plots of some typical steady gap-solitons (as described above) are now presented. Figures 3 and 4 show the real and imaginary parts of the amplitudes $\hat{A}$ and $\hat{B}$ on the lower and upper interfaces, respectively, for $\hat{\sigma} = 1.01$ and $\hat{\sigma}' = 0$ (see Table 1) falling into the "+"-case. The solitons (elevation ones) are clearly in phase.

Then, in Figures 5 and 6 we show the analogous plots for the case $\hat{\sigma} = 1.50$ and $\hat{\sigma}' = 0.40$ (see Table 2) falling into the "$-$"-case. The solitons (one depression and one elevation) are clearly now out of phase. An important feature of these gap-solitons is that, unlike the usual solitons of the NLS equation, there is a significant phase variation, particularly noticeable in Figures 5 and 6. That is, the total spatial wavenumber is $k + \phi', k + \psi'$ for A, B respectively, where from (44, 45) we se that the terms ϕ', ψ' are proportional to R^2.

$\hat{\sigma}$	$\hat{U}_1$	$\hat{V}_1$	sign($\tilde{W}$)	valid
0	∓ 0.0141	∓ 0.0070	$-$	N
0.20	∓ 0.0154	∓ 0.0051	$-$	N
0.40	∓ 0.0167	∓ 0.0036	$-$	N
0.51	± 0.0173	∓ 0.0028	$+$	N
0.99	∓ 0.0199	∓ 0.00005	$+$	N
1.01	∓ 0.0200	± 0.00005	$-$	Y
1.35	∓ 0.0216	± 0.0016	$-$	Y
1.38	∓ 0.0217	± 0.0017	$+$	Y
1.50	∓ 0.0222	± 0.0022	$+$	Y
2.00	∓ 0.0244	± 0.0041	$+$	Y
2.50	∓ 0.0263	± 0.0056	$+$	Y
3.30	∓ 0.0292	± 0.0078	$+$	Y

Table 2: As for Table 1, except that now $\hat{\sigma}' = 0.40$.

5 Conclusion

We recall from section 1 that the general aim here is, for any specific fluid flow system, to determine the presence of the linear resonance conditions where there are two independent wave modes at the resonance point, then to determine the resulting envelope system of equations, and finally to demonstrate that this system supports gap-soliton solutions. For the three-layered fluid flow described in section 2 (see Figure 2) the envelope system was derived by GC, and we have presented a summary of that analysis in section 2 leading to the equation system (21, 22). Then in section 3 we have exhibited explicitly the gap-soliton solutions of that system. Particular interest attaches to the case when the envelope and carrier speeds are identical, so that gap-soliton is then a steady travelling-wave solution of the full fluid system. This case was discussed in section 4, where we have shown that this three-layered fluid model can exhibit a large class of such gap-soliton solutions. Although the existence of gap-solitons is well-known in other branches of phsyics, such as nonlinear optics, we believe that is the first time they have been found in a fluid flow.

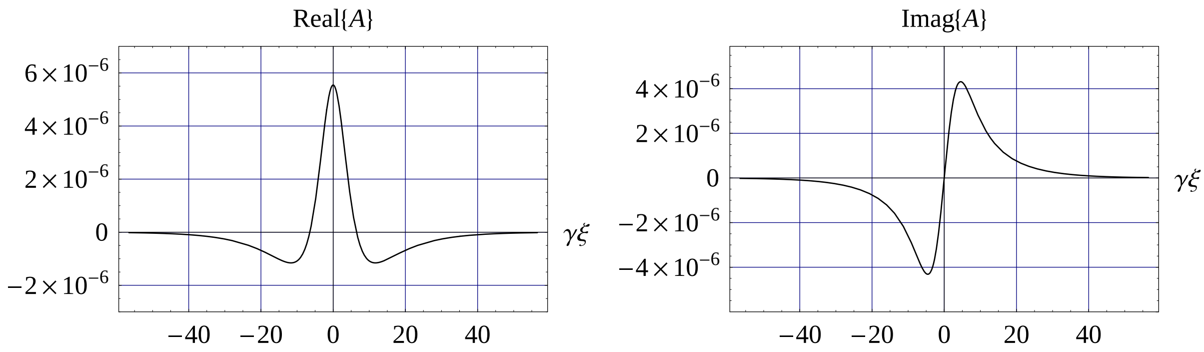

Figure 3: Steady gap-solitons for $\hat{\sigma} = 1.01$, $\hat{\sigma}' = 0$ when $k = 1m^{-1}$, $H_2 = 50m$, $H_1 = H_3 = 500m$ on the lower interface ($\tilde{W} < 0$).

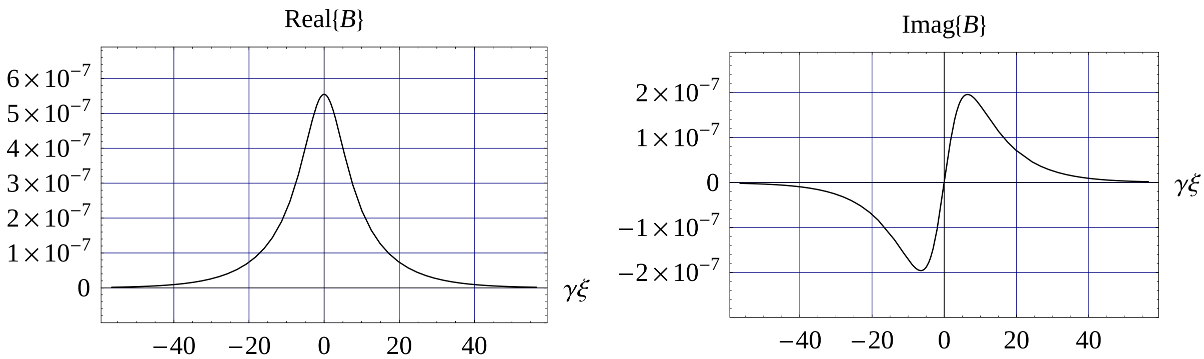

Figure 4: As for Figure 3, on the upper interface.

A Appendix

From GC, the quantities D_1, D_2, E are given by

$$D_1(\omega, k) = g(\rho_1 - \rho_2) + \sigma k^3 - \{\frac{\rho_1(\omega - kU_1)^2}{kT_1} + \frac{\rho_2(\omega - kU_2)^2}{kT_2}\}\,, \tag{56}$$

$$D_3(\omega, k) = g(\rho_2 - \rho_3) + \sigma' k^3 - \{\frac{\rho_2(\omega - kU_2)^2}{kT_2} + \frac{\rho_3(\omega - kU_3)^2}{kT_3}\}\,, \tag{57}$$

$$E(\omega, k) = \frac{\rho_2(\omega - kU_2)^2}{kS_2}\,, \tag{58}$$

where $T_i = \tanh kH_i$ and $S_i = \sinh kH_1$ for $i = 1, 2, 3$. At resonance, these give

$$D_{1\omega} = \frac{2\rho_1 U_1}{T_1}\,, \qquad D_{3\omega} = \frac{2\rho_3 U_3}{T_3}\,, \tag{59}$$

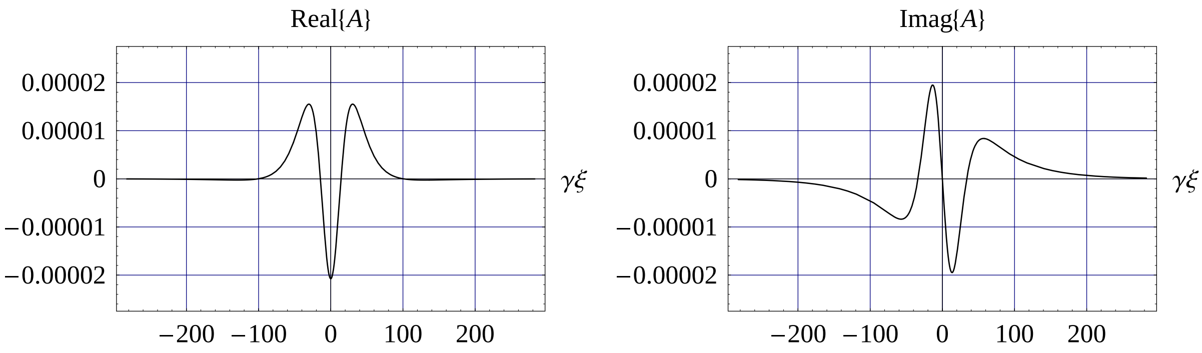

Figure 5: Steady gap-solitons for $\hat{\sigma} = 1.50$, $\hat{\sigma}' = 0.40$ when $k = 1m^{-1}$, $H_2 = 50m$, $H_1 = H_3 = 500m$ on the lower interface ($\tilde{W} > 0$).

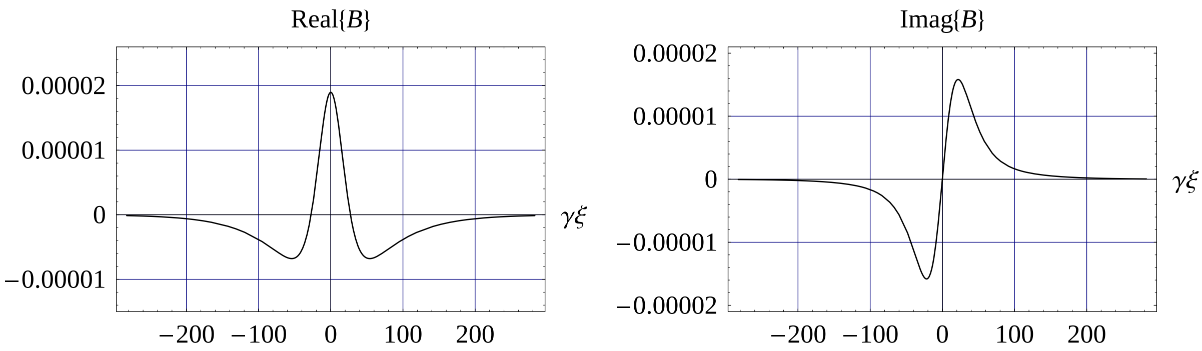

Figure 6: As for Figure 5, on the upper interface.

The group velocities at the resonant wavenumber are given by

$$V_1 = \frac{1}{2}U_1\{1 - \frac{kH_1T_1}{S_1^2}\} - \frac{\sigma k T_1}{\rho_1 U_1}\,, \tag{60}$$

$$V_3 = \frac{1}{2}U_3\{1 - \frac{kH_3T_3}{S_3^2}\} - \frac{\sigma' k T_3}{\rho_3 U_3}\,. \tag{61}$$

.

The mean flow variables satisfy the equation set

$$\bar{\eta}_T + U_1\bar{\eta}_X + H_1\bar{u}_{1X} = \frac{2kU_1}{T_1}|A|_X^2\,, \tag{62}$$

$$\bar{\eta}_T - C = 0\,, \tag{63}$$

$$\bar{\zeta}_T + H_2\bar{u}_{2X} - C = 0\,, \tag{64}$$

$$\bar{\zeta}_T + U_3\bar{\zeta}_X + H_3\bar{u}_{3X} = -\frac{2kU_3}{T_3}|B|_X^2\,, \tag{65}$$

obtained from averaging the kinematic boundary conditions at each interface, while the variable C is related to the mean vertical velocity in the middle layer. The system is closed by averaging the pressure boundary conditions at each interface to get

$$\rho_1(\bar{u}_{1T} + U_1\bar{u}_{1X}) - \rho_2\bar{u}_{2T} + g(\rho_1 - \rho_2)\bar{\eta}_X = -\rho_1 U_1^2 k^2 \frac{1 - T_1^2}{T_1^2}|A|_X^2, \tag{66}$$

$$\rho_2\bar{u}_{2T} - \rho_3(\bar{u}_{3T} + U_3\bar{u}_{3X}) + g(\rho_2 - \rho_3)\bar{\zeta}_X = \rho_3 U_3^2 k^2 \frac{1 - T_3^2}{T_3^2}|B|_X^2 . \tag{67}$$

The remaining coefficients in the evolution equations (21, 22) are given by

$$\mu_1 = \frac{1}{D_1(2\omega, 2k)}\{-\rho_1^2 U_1^4 k^4 \frac{9 + 4T_1 + 4T_1^3 - T_1^4}{2T_1^4} + 2\rho_1 U_1^2 k^3 [g(\rho_1 - \rho_2) + 4\sigma k^2]\frac{1}{T_1^2}\}$$
$$-2\rho_1 U_1^2 k^3 \frac{1 + T_1 - 2T_1^2}{T_1^3} - \frac{3}{2}\sigma k^4 , \tag{68}$$

$$\mu_3 = \frac{1}{D_3(2\omega, 2k)}\{-\rho_3^2 U_3^4 k^4 \frac{9 + 4T_3 + 4T_3^3 - T_3^4}{2T_3^4} + 2\rho_3 U_3^2 k^3 [g(\rho_2 - \rho_3) + 4\sigma' k^2]\frac{1}{T_3^2}\}$$
$$-2\rho_3 U_3^2 k^3 \frac{1 + T_3 - 2T_3^2}{T_3^3} - \frac{3}{2}\sigma' k^4 , \tag{69}$$

$$\nu_1 = \rho_1 U_1^2 k^2 \frac{1 - T_1^2}{T_1^2} , \quad \nu_3 = -\rho_3 U_3^2 k^2 \frac{1 - T_3^2}{T_3^2} . \tag{70}$$

Finally, the coefficients arising in the solitary-wave solution of the mean flow equations are given by

$$\eta_A = -\rho_1[U_1^2 k^2 \frac{1 - T_1^2}{T_1^2} + (U_1 - V)\frac{2kU_1}{T_1 H_1}][g(\rho_2 - \rho_3) + \rho_3\frac{(U_3 - V)^2}{H_3} - \rho_2\frac{V^2}{H_2}]\frac{1}{d} , \tag{71}$$

$$\eta_B = -\rho_2\rho_3\frac{V^2}{H_2}[U_3^2 k^2 \frac{1 - T_3^2}{T_3^2} - (U_3 - V)\frac{2kU_3}{T_3 H_3}]\frac{1}{d} , \tag{72}$$

$$\zeta_A = \rho_1\rho_2\frac{V^2}{H_2}[U_1^2 k^2 \frac{1 - T_1^2}{T_1^2} + (U_1 - V)\frac{2kU_1}{T_1 H_1}]\frac{1}{d} , \tag{73}$$

$$\zeta_B = \rho_3[U_3^2 k^2 \frac{1 - T_3^2}{T_3^2} - (U_3 - V)\frac{2kU_3}{T_3 H_3}][g(\rho_1 - \rho_2) - \rho_1\frac{(U_1 - V)^2}{H_1} - \rho_2\frac{V^2}{H_2}]\frac{1}{d} , \tag{74}$$

$$u_{1A} = \frac{1}{\rho_1(U_1 - V)}[(\rho_2\frac{V^2}{H_2} - g(\rho_1 - \rho_2))\eta_A - \rho_2\frac{V^2}{H_2}\zeta_A - \rho_1 U_1^2 k^2 \frac{1 - T_1^2}{T_1^2}] , \tag{75}$$

$$u_{1B} = \frac{1}{\rho_1(U_1 - V)}[(\rho_2\frac{V^2}{H_2} - g(\rho_1 - \rho_2))\eta_B - \rho_2\frac{V^2}{H_2}\zeta_B] , \tag{76}$$

$$u_{3A} = \frac{1}{\rho_3(U_3 - V)}[(-\rho_2\frac{V^2}{H_2} - g(\rho_2 - \rho_3))\zeta_B + \rho_2\frac{V^2}{H_2}\eta_B - \rho_3 U_3^2 k^2 \frac{1 - T_3^2}{T_3^2}] , \tag{77}$$

$$u_{3B} = \frac{1}{\rho_3(U_3 - V)}[(-\rho_2\frac{V^2}{H_2} - g(\rho_2 - \rho_3))\zeta_A + \rho_2\frac{V^2}{H_2}\eta_A] , \tag{78}$$

$$d = [g(\rho_1 - \rho_2) - \rho_1\frac{(U_1 - V)^2}{H_1} - \rho_2\frac{V^2}{H_2}][g(\rho_2 - \rho_3) + \rho_3\frac{(U_3 - V)^2}{H_3} - \rho_2\frac{V^2}{H_2}] - \rho_2^2\frac{V^4}{H_2^2} . \tag{79}$$

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