Research Article Open Access

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Analysis of Smart Piezo-Magneto-Thermo-Elastic Composite and Reinforced Plates: Part I – Model Development

Abstract: A comprehensive micromechanical model for the analysis of a smart composite piezo-magneto-thermo-elastic thin plate with rapidly-varying thickness is developed in the present paper. A rigorous three-dimensional formulation is used as the basis of multiscale asymptotic homogenization. The asymptotic homogenization model is developed using static equilibrium equations and the quasi-static approximation of Maxwell's equations. The work culminates in the derivation of a set of differential equations and associated boundary conditions. These systems of equations are called unit cell problems and their solution yields such coefficients as the effective elastic, piezoelectric, piezomagnetic, dielectric permittivity and others. Among these coefficients, the so-called product coefficients are also determined which are present in the behavior of the macroscopic composite as a result of the interactions and strain transfer between the various phases but can be absent from the constitutive behavior of some individual phases of the composite material. The model is comprehensive enough to allow calculation of such local fields as mechanical stress, electric displacement and magnetic induction. In part II of this work, the theory is illustrated by means of examples pertaining to thin laminated magnetoelectric plates of uniform thickness and wafer-type smart composite plates with piezoelectric and piezomagnetic constituents. The practical importance of the model lies in the fact that it can be successfully employed to tailor the effective properties of a smart composite plate to the requirements of a particular engineering application by changing certain geometric or material parameters. The results of the model constitute an important refinement over previously established work. Finally, it is shown that in the limiting case of a thin elastic plate of uniform thickness the derived model converges to the familiar classical plate model.

Keywords: smart composite piezo-magneto-thermo-elastic thin plate; asymptotic homogenization; effective properties; product properties

DOI 10.2478/cls-2014-0002
Received August 7, 2014 ; accepted September 11, 2014

1 Introduction

Significant advancements in the production and application of composites coupled with emerging technologies in the fields of sensors and actuators have permitted the integration of smart composites in an increasingly larger number of engineering applications. Of particular interest among smart composites is the class of structures which include both piezoelectric and piezomagnetic phases. The strain transfer and general interactions between the various phases of these composites give rise to the so-called product properties, see Newnham et al. [1]. These properties are found in the behavior of the macroscopic composite but are usually absent from the constituent behavior of the individual phases. Examples of product properties are

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the magnetoelectric, pyromagnetic and pyroelectric properties, see Nan et al. [2], Bichurin et al. [3]. The magnetoelectric property is the behavior which governs the generation of an electric displacement when a magnetic field is applied and vice-versa. In particular, applying a magnetic field induces a mechanical strain in the piezomagnetic phase. In turn, provided that there is satisfactory degree of bonding between the two constituents, this magnetically induced strain is transferred to the piezoelectric phase which then produces an electric field. Likewise, the pyroelectric and pyromagnetic product properties refer to the generation of an electric or magnetic field when a thermal load is applied.

The unique properties of magnetoelectric composites render them suitable candidates for a broad range of novel practical applications in the form of components, devices and systems. For example their sensitivity to external stimuli (electric and magnetic fields, temperature etc.) can be exploited for frequency tunable devices such as resonators and filters, magnetic field sensors, energy harvesting transducers, miniature antennas, etc. [3–9]. Other attractive potential applications of some classes of magnetoelectric composites include data storage devices and spintronics [10], biomedical sensors for EEG/MEG and other relevant equipment [11, 12] etc. In view of the aforementioned (and many more) practical applications, the main objective of this work is to develop accurate micromechanical models that can be used to design magnetoelectric and general smart composite and reinforced plates. The model must be comprehensive enough to afford the designer significant flexibility with regards to both the structural make-up and the overall geometry of the given structures.

The use of composites and smart composites in new engineering applications is often limited due to the lack of reliable data concerning their long-term behavior. This disadvantage could be successfully mitigated if the behavior of such structures could be determined at the design stage. This can be effectively achieved via the development of accurate micromechanical models. To be useful, these models must be comprehensive enough to capture all the important behavioral characteristics of the composite structure. At the same time, they must not be too complicated to be used effectively, efficiently and expediently. Ideally, and in order to be readily amenable to design, such models should lead to closed-form expressions for the determination of effective properties in terms of the material and geometric parameters of the constituents and the macroscopic structure.

Despite the increased interest in the magnetoelectric effect and other product properties, little research work pertaining to the micromechanical modeling of this behavior exists. As expected, both analytical and numerical (principally finite element-based) approaches have been examined and implemented. Noteworthy among the analytical models are the works of Harshe et al. [13, 14] and Avellaneda and Harshe [15], Huang et al. [16–19], Bichurin et al. [20, 21], Soh and Liu [22], Bravo-Castillero et al. [23], Ni et al. [24] Akhabarzadeh et al. [25], and others. Harshe et al. [13, 14] and Avellaneda and Harshe [15] obtain the magnetoelectric coefficients of 2-2 piezoelectric/magnetostrictive multilayer composites for mechanically free and clamped structures. Huang and Kuo [16] developed a comprehensive model pertaining to piezoelectric/piezomagnetic composites on the basis of the classical works of Eshelby [26] and Mori-Tanaka [27]. In particular, their model incorporated reinforcements in the form of ellipsoidal inclusions which allowed the reinforcement geometry to vary from thin flakes to long continuous fibers. The determination of Eshelby-like tensors [26] allowed the authors to compute not only the effective properties, but also the local fields around the inclusions and the pertinent stress concentration factors. Using the Mori-Tanaka approach [27] the interactions between the constituent phases were also examined. In an extension of this work, Huang [17] obtained closed-form solutions for a transversely isotropic matrix and reinforcements in the shape of elliptic cylinders, circular cylinders, disks and ribbons. The resulting expressions are functions of the inclusion properties and geometry, as well as the pertinent volume fractions. In another work, Huang et al. [18] examined the magnetoelectric effect in piezoelectric/piezomagnetic bilayers under coupled bending and stretching loading conditions. They discovered that the magnetoelectric coupling coefficients in this case were significantly higher than in the case of pure stretching. In an interesting new study based on their earlier works, Huang et al. [19] obtained the magnetoelectric coefficients in composites of continuous piezoelectric fibers embedded in a piezomagnetic matrix. The authors also obtained an analytical expression for the optimized fiber volume fraction for maximizing the magnetoelectric coupling coefficients. Surprisingly, their results indicated that the optimum volume fraction is a function of the elastic properties of the constituents and is independent of the magnetic and electric properties.

Bichurin et al. [20] investigated the magnetoelectric effect in ferromagnetic/piezoelectric multilayer composites using a two-step approach. In the first step they wrote down the constitutive relationships of the individual phases and in the second they used the corresponding expressions of the macroscopic composite. The same authors, Bichurin et al [21], extended their work to magnetoelectric nanocomposites. Soh and Liu [22] adopted a new approach in deriving eight sets of constitutive equations characterizing magnetoelectric composites directly from eight thermodynamic potentials. The theoretical framework of their study is important in that it established the necessary relationships that must exist between the various material constants that appear in the constitutive laws. Bravo-Castillero et al. [23] used generalized test functions to avoid singularities that occur due to the discontinuity at the interphase between the constituents. To illustrate their model they obtained closed-form expressions for the effective properties of piezoelectric/piezomagnetic laminates. Ni et al. [24] investigated the magnetoelectric properties of 3-ply polycrystalline multiferroic laminates consisting of a piezoelectric lamina sandwiched between two ferromagnetic ones. In their modeling approach, the authors applied a magnetic field and determined the induced electric displacement. They computed the magnetoelectric coupling coefficients as ratios of applied magnetic field to induced polarization. Their work showed that the magnetoelectric constants depend strongly on the orientation of the magnetic fields. Akbarzadeh et al. [25] considered, among others, the pyroelectric coefficients when analyzing the thermo-electro-magneto-elastic behavior of rotating functionally graded piezoelectric cylinders. Other works can be found in Kirchner et al. [28], Pan and Heyliger [29], Benveniste [30], Nan et al. [31], Spyropoulos et al. [32] and others.

Naturally, the finite element technique has proven a popular method for analyzing magnetoelectric composites. In this respect, special consideration must be given to the works of Tang and Yu [33, 34], who employed the variational asymptotic method to investigate periodic two-phase and three-phase structures. Starting from the total electromagnetic enthalpy or thermodynamic potential, the authors then applied constraint minimization. The pertinent equations were solved using the finite element technique. The authors illustrated their model by considering two types of fiber-reinforced composites; one consisted of piezoelectric fibers embedded in a piezomagnetic matrix and the other of piezoelectric and piezomagnetic fibers embedded in an elastic matrix. Their calculated effective coefficients agreed well with other reported values. Sunar et al. [35] used the finite element method to examine piezoelectric/piezomagnetic composites. The authors began by defining two energy functionals and then applied Hamilton's principle to derive the constitutive equations for the smart structure. The authors then employed a finite element approach to a barium titanate/cobalt ferrite two-layer composite. In particular they examined the generation of a magnetic field when an applied electrostatic field induces a piezoelectric mechanical strain. The authors' results conformed fairly well to those obtained via a simple analytical technique. Other work can be found in Lee et al. [36], Liu et al. [37], Mininger et al. [38], Sun et al. [39] and others.

The micromechanical modeling of periodic composites and smart composites is characterized by rapidly varying material coefficients with period "ε", the characteristic dimension of the periodicity or unit cell. At the same time, the dependent local fields such as mechanical stress, magnetic induction and electric displacement are functions of both periodic (microscopic) and non-periodic (macroscopic) variables. The coupling of the microscopic and macroscopic scales renders even the numerical analysis of the aforementioned structures rather cumbersome. Further, an analytic solution is unattainable in all but the simplest geometries. These problems could be overcome if the two scales were decoupled and each handled separately. An effective technique which can be used to achieve precisely this is that of asymptotic homogenization. The mathematical framework of asymptotic homogenization can be found in Bensoussan et al. [40], Sanchez-Palencia [41], Bakvalov and Panasenko [42] and Cioranescu and Donato [43]. Many problems in elasticity, thermoelasticity and piezoelasticity have been solved via asymptotic homogenization. Examples can be found in Kalamkarov [44], Kalamkarov and Kolpakov [45], Kalamkarov and Georgiades [46], Georgiades et al. [47], Hassan et al. [48], Saha et al. [49], Guedes and Kikuchi [50], Sevostianov and Kachanov [51] and many others.

Currently, the preponderance of uses of composite materials is in the form of plate and shell structural members, the strength and reliability of which, combined with reduced weight and concomitant material savings, offer the designer very impressive possibilities in many applications. It often happens that the reinforcing elements such as fibers form a regular array with a period much smaller than the characteristic dimension of the composite structure; consequently asymptotic homogenization analysis is applicable.

Homogenized models of plates with periodic non-homogeneities in tangential coordinates have been developed in this way by Duvaut [52], Duvaut and Metellus [53], Adrianov and Manevitch [54], Adrianov et al. [55] and others. Particularly noteworthy is the modified technique employed by Caillerie [56, 57] in his conduction studies. In particular, two sets of microscopic variables were introduced, one of which pertained to the tangential directions (characterized by periodicity) while the other variable related to the transverse direction in which no periodicity exists. Kohn and Vogelius, [58, 59] adopted this approach in their study of the pure bending of a thin, linearly elastic homogeneous plate.

Kalamkarov [44], developed general homogenized composite shell models by applying the modified two-scale asymptotic method directly to three-dimensional elastic and thermoelastic problems for a thin curvilinear composite layer with rapidly varying thickness. Challagulla et al. [60] employed this methodology to develop rigorous asymptotic homogenization models for thin smart composite shells and illustrated their results by means of interesting and practically important examples including single-walled carbon nanotubes. Kalamkarov and Kolpakov [61] developed a new model for the analysis of clamped piezoelastic plates. Hadjiloizi et al. [62] implemented a general model (based on the time-varying form of Maxwell's equations and the dynamic force balance) for the micromechanical dynamic analysis of magnetoelectric thin plates with rapidly varying thickness. In [62] however, only an in-plane temperature variation is taken into consideration and therefore any out-of-plane thermal effects are ignored. Thus, unlike in the present work, the out-of-plane thermal expansion, pyroelectric and pyromagnetic coefficients were not captured in [62]. As a further consequence, in the general field expressions for the force and moment resultants, electric and magnetic fields etc. the influence of the out-of-plane temperature variation is neglected. More importantly however, the micromechanical model in [62] was only applied to simple laminated plates. One can certainly not argue against the practical significance of such structures; on the other hand it is evident that laminates offer little design flexibility with respect to geometry which limits their application potential. In contrast, the model developed in the present paper allows for explicitly different periodicity in the tangential directions of the structure. This feature makes the model much more amenable to the analysis and design of not only laminated plates, but also reinforced plates such as wafer- and rib-reinforced structures shown in Section 7. Such structures were not at all considered in [62].

Also relevant to the present papers are the works of Hadjiloizi et al. [63, 64] and Kalamkarov and Georgiades [65], Georgiades and Kalamkarov [66]. In [63, 64], Hadjiloizi et al. developed two general three-dimensional models for magnetoelectric composites. One model used dynamic force and thermal balance and the time-varying form of Maxwell's equations to determine closed-form expressions for the effective properties. The second model used the quasi-static approximation of the aforementioned constitutive equations. However, these models are three-dimensional in nature and as such cannot capture the mechanical, thermal, etc. behavior that is related to bending, twisting and general out-of-plane deformation as well as electric and magnetic field generation. The model developed in the current work and its companion paper [67] however, accomplishes precisely this; it employs a modified asymptotic homogenization technique, which makes use of two sets of microscopic variables (and is therefore quite different than the "classical" schemes of [63], [64]) that permit the decoupling of the in-plane and out-of plane behavior of the structure under consideration. For example, the elastic coefficients are separated into the familiar extensional, bending and coupling coefficients. This is not possible to achieve with the 3D models in [63] and [64]. Essentially, the two models are applicable to entirely different geometries. The 3D models in [63, 64] can be used to analyze structures of comparable dimensions (such as *thick* laminates) but cannot be used for thin structures such as wafer- and rib-reinforced plates. The micromechanical models developed in the current works however are applicable to structures with a much smaller dimension in the transverse direction than in the other two directions. Thus, it can be used in the design and analysis of composite and reinforced plates such as the aforementioned wafer- and rib-reinforced structures (see Section 7), three-layered honeycomb-cored magnetoelectric plates, *thin* laminates etc.

In [65] and [66], Kalamkarov and Georgiades performed only a semi-coupled analysis of a composite or reinforced plate and therefore the resulting expressions of the effective coefficients do not reflect the influence of such parameters as the electric permittivity, magnetic permeability, primary magnetoelectricity etc. The present work and its companion paper [67] however, perform a fully coupled analysis and as a consequence the expressions for the effective coefficients involve all pertinent material parameters. As an example, the effective elastic coefficients depend not only on the elastic properties of the constituent materials, but also on the piezoelectric, piezomagnetic, magnetic permeability, dielectric permittivity and other parameters. The same holds true for the remaining effective coefficients. In a sense, the thermoelasticity, piezoelectricity and piezomagnetism problems are entirely coupled and the solution of one affects the solutions of the others. This feature is captured in the present papers, but not in previously published works, such as [65] and [66]. Thus, the results presented here represent an important refinement of previously established results. To the authors' best knowledge, this is the first time that completely coupled piezo-magneto-thermo-elastic effective coefficients for reinforced plates have been presented and analyzed.

In view of the practical applications mentioned earlier on in this Section, the primary importance of this work lies in the fact that it develops a novel micromechanical model that leads to closed-form design-oriented equations easily integrated in MATLAB^{TM} or other similar software packages. These equations can be used to analyze and design magnetoelectric and other smart composite and reinforced plates with a broad range of geometries. The results of the developed models show improved accuracy as compared to previously published results.

In particular, the present paper deals with the development of appropriate micromechanical models to examine the quasi-static plane stress solution of the aforementioned magnetoelectric composite and reinforced plates. The work is implemented in two parts. In part II [67] the derived models are applied to the practically important cases of thin magnetoelectric laminates and wafer-reinforced magnetoelectric plates. Our overall objectives are to: (a) obtain expressions for the dependent field variables, (b) derive closed-form expressions for all effective coefficients including product properties, (c) compare the results of our model with those of other models and illustrate, where possible, the improvements over previously established results. Following this introduction, the basic relations of the three-dimensional problem are formulated in Section 2 and the two-scale asymptotic expansions for the displacement and stress fields are introduced in Section 3. The equilibrium equations and pertinent boundary conditions are derived in Section 4 followed by the determination of the unit cell problems in Section 5. The governing equations and the effective coefficients of the homogenized plate are obtained in Section 6 in which a comparison of the homogenized plate with a smart composite lamina is also performed. Section 7 is a brief overview of the structures to be considered in Part II [67] of this work, and finally section 8 concludes this paper.

2 Problem Formulation

Consider a thin smart layer representing an inhomogeneous solid with wavy surfaces, containing a large number of periodically arranged actuators as shown in Fig. 1. This periodic structure is obtained by repeating a certain small unit cell Ω_δ in the $x_1 - x_2$ plane (Fig. 1). In the parlance of asymptotic homogenization the unit cell is thus sometimes referred to as a "periodicity cell", [46–48]. All three pertinent coordinates are assumed to have been made dimensionless by division by a certain characteristic dimension of the body, L. Note that the shape of the top and bottom surfaces of the layer is determined by the type of the surface reinforcement (for example by the shape of stiffeners or reinforcing ribs) or actuator (when the actuator is surface attached and not embedded within the structure). The surfaces can be plane if surface reinforcements or actuators are not used. In the context of the present work, the meaning of actuator (piezoelectric, piezomagnetic, magnetostrictive etc.) is a device that can be used to induce stresses and strains in a coordinated fashion [64–66].

Furthermore, it is assumed that stress concentrations and/or property variations at the interphase region between the matrix and the reinforcements and/or actuators are negligible. Essentially, it is assumed that the interphase regions are highly localized and do not contribute significantly to the integral over the entire unit cell domain. In practical terms, the error incurred will be negligible if the dimensions of the actuators/reinforcements are much smaller than the spacing between them. As an indication, we note that for the purely elastic case, Kalamkarov [44] showed that if the spacings between the unit cells are at least ten times bigger than the thickness of the reinforcements then the error in the values of the effective elastic coefficients incurred by ignoring the regions of overlap between the reinforcements is less than 1%.

The unit cell of the problem is defined by the following inequalities (see Fig. 1),

$$\left\{-\frac{\delta h_1}{2} < x_1 < \frac{\delta h_1}{2}, \quad -\frac{\delta h_2}{2} < x_2 < \frac{\delta h_2}{2}, S^- < x_3 < S^+\right\}, \quad \text{where} \quad S^\pm = \pm\frac{\delta}{2} \pm \delta F^\pm\left(\frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right) \tag{2.1}$$

and the microscopic behavior of this smart structure is characterized by means of the following boundary value problem:

$$\frac{\partial \sigma_{ij}\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right)}{\partial x_j} = P_i\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right) \tag{2.2a}$$

$$\frac{\partial D_i\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right)}{\partial x_i} = 0 \tag{2.2b}$$

$$\frac{\partial B_i\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right)}{\partial x_i} = 0 \tag{2.2c}$$

As well, the irrotational electric and magnetic (in the absence of free conduction currents) fields may be written down as the gradients of scalar functions, φ and ψ.

$$E_i\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right) = -\frac{\partial \varphi\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right)}{\partial x_i} \tag{2.2d}$$

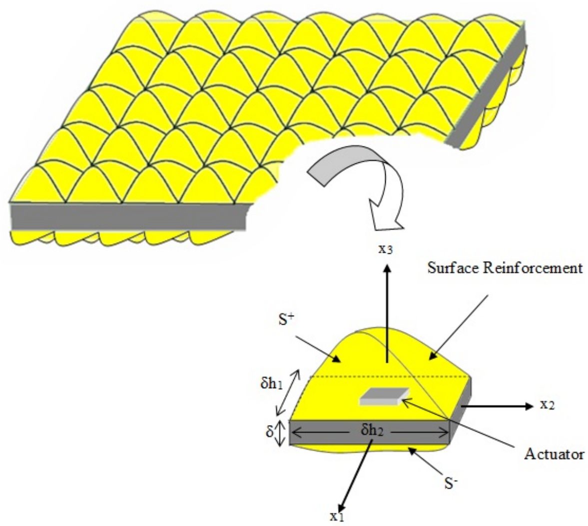


Figure 1: Thin smart composite plate with rapidly varying thickness and its periodicity cell.

$$H_i\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right) = -\frac{\partial \psi\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right)}{\partial x_i} \tag{2.2e}$$

In Eqs. 2.2a- 2.2e, σ_{ij} is the mechanical stress, D_i and B_i are, respectively, the electric displacement and magnetic induction, E_i and H_i are the electric and magnetic fields and P_i represents a generic body force. Eq. 2.2a represents the static equilibrium equations and Eqs. 2.2b and 2.2c represent the quasi-static approximation of Maxwell's Equations. It should be noted that all field variables defined thus far are characterized by both periodic (dependence on $x_i/\delta h_i$) and non-periodic components (dependence on x_i) as is expected for the periodic structure of Fig. 1, see for example Kalamkarov and Georgiades [65].

We will further assume that the top and bottom surfaces of the plate, $S^{\pm}$, have the following boundary conditions:

$$\sigma_{ij} n_j = p_i, \quad D_i n_i = 0, \quad B_i n_i = 0, \quad on\ S^{\pm} \tag{2.3a}$$

where for the surfaces $x_3 = S^{\pm}(x_1, x_2)$ we have the following unit normal vector, see Kalamkarov [44]:

$$n^{\pm} = \left(\mp\frac{\partial S^{\pm}}{\partial x_1}, \mp\frac{\partial S^{\pm}}{\partial x_2}, 1\right)\left[\left(\frac{\partial S^{\pm}}{\partial x_1}\right)^2 + \left(\frac{\partial S^{\pm}}{\partial x_2}\right)^2 + 1\right]^{-1/2} \tag{2.3b}$$

The first expression in 2.3a is the familiar Cauchy's Law, the second implies that we have no free surface electrical charge and the third indicates that the normal component of the magnetic induction field is continuous at the top surface. In Eq. 2.3b, p_i (not to be confused with body forces, P_i) represents the surface tractions (external forces per unit area acting on the top and bottom surfaces of the plate, see Kalamkarov [44]). On the lateral surfaces we will assume the following boundary conditions (where u_i is the mechanical displacement):

$$u_i = 0, \quad \varphi = \delta^2 e\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right),$$
$$\psi = \delta^2 h\left(x_1, x_2, x_3, \frac{x_1}{\delta h_1}, \frac{x_2}{\delta h_2}\right) \tag{2.3c}$$

The boundary value problem of Eqs. 2.2a – 2.2c must be complemented by the appropriate constitutive equations in the form of:

$$\begin{aligned}\sigma_{ij}\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right) =& C_{ijkl}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial u_k\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_l} \\ &+ e_{kij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial \varphi\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_k} \\ &+ Q_{kij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial \psi\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_k} \\ &- \delta\Theta_{ij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right) T\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)\end{aligned} \tag{2.4a}$$

$$\begin{aligned}D_i\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right) =& \delta\left\{e_{ijk}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial u_k}{\partial x_l}\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)\right. \\ &-\varepsilon_{ij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial \varphi\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_k} \\ &-\lambda_{ij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial \psi\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_k} \\ &\left.+\delta\xi_i\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right) T\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)\right\}\end{aligned} \tag{2.4b}$$

$$\begin{aligned}B_i\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right) =& \delta\left\{Q_{ijk}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial u_k}{\partial x_l}\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)\right. \\ &-\lambda_{ij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial \varphi\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_k} \\ &-\mu_{ij}\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right)\frac{\partial \psi\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)}{\partial x_k} \\ &\left.+\delta\eta_i\left(x_3, \frac{x_\alpha}{\delta h_\alpha}\right) T\left(x_i, \frac{x_\alpha}{\delta h_\alpha}\right)\right\}\end{aligned} \tag{2.4c}$$

Here, $e_{kl} = \partial u_k/\partial x_l$ is the second order strain field, and C_{ijkl}, e_{ijk}, Q_{ijk}, and Θ_{ij} are the tensors of the elastic, piezoelectric, piezomagnetic and thermal expansion coefficients respectively. Finally, $\varepsilon_{ij}, \lambda_{ij}, \mu_{ij}, \xi_i$ and η_i represent, respectively, the dielectric permittivity, the magnetoelectric, the magnetic permeability, the pyroelectric and the pyromagnetic tensors. We reiterate that as a consequence of the fact that the composite layer is periodic only in the tangential directions the material parameters are dependent on $x_\alpha/\delta_{h\alpha}$ and x_3 while the dependent field variables are also dependent on $x_\alpha = (x_1, x_2)$. Eqs. 2.4b and 2.4c show that the constituents of the structure under investigation may, if desired, exhibit magnetoelectric, pyroelectric and pyromagnetic characteristics; Newnham et al. [1] refer to this as the "primary" effect.However, it is more likely that these product properties only appear in the behavior of the macroscopic composite as a consequence of the interactions between the various phases as explained in the previous section. In Eq. 2.4a and in the sequel Roman letters, i, j, k, ... will vary from 1 to 3, while their Greek counterparts, $\alpha, \beta, \gamma, \ldots$ will assume values of 1 or 2 only.

3 Asymptotic analysis and basic assumptions

The overall objective of this paper is to obtain general expressions for the effective coefficients (including product properties) and the dependent field variables for magnetoelectric thin plates with rapidly varying thickness. In order to be able to obtain the aforementioned expressions in a form that is immediately comparable to other works, and, more importantly, readily applicable to composite and reinforced structures of different geometrical and compositional make-up, the following procedure will be adhered to in Sections 3-6 of this work. We begin with the asymptotic expansions of all field variables of interest in terms of the dimensionless thickness of the plate δ. Realizing that we can obtain all desirable information via the determination of the mechanical displacement and electric and magnetic potentials, we recast each of the three governing equations 2.2a – 2.2c and associated boundary conditions 2.3a as functions of the leading terms of the asymptotic expansions of the mechanical stress, electric displacement and magnetic induction. The resulting three equations/boundary conditions are then expressed in terms of the desired mechanical displacement, u_i, and electric and magnetic potentials, φ and ψ respectively. The mathematical form of these latter expressions is such that it permits us to write down the solutions for u_i, φ and ψ as linear combinations involving a set of so-called local functions. The independence of the local functions allows us to group them together into different sets of problems called unit cell problems which are solved entirely on the domain of the unit cell and are independent of the global formulation of the original problem. The solution of the unit cell problems eventually yields the effective or homogenized coefficients, after application of an associated averaging or homogenization procedure. At this stage, we have managed to essentially smooth out the sub-structural variations that exist in the original inhomogeneous composite plate and generate an equivalent homogeneous structure characterized by a single set of material parameters called effective coefficients. Once these effective coefficients are determined, a wide variety of boundary value problems involving a given composite geometry can be studied with relative ease. It would not be remiss to mention at this point that this methodology or some variant thereof has been followed in many of the authors' previous works, see for example [44–49, 60–65].

It is apparent from the preceding analysis that the smart composite structure under consideration is characterized by two scales; the microscopic scale which is a manifestation of periodicity in the tangential directions, and the macroscopic scale which arises from the global formulation of the problem, see Kalamkarov [44], Challagulla et al [60]. To this end, we begin our analysis with the introduction of the microscopic or "fast" variables,

$$y_1 = \frac{x_1}{\delta h_1}, \quad y_2 = \frac{x_2}{\delta h_2}, \quad z = \frac{x_3}{\delta} \tag{3.1}$$

remembering that δ is the thickness of the smart layer. Hence, in terms of these variables, the unit cell Ω_δ is defined by

$$\left\{-\frac{1}{2} < y_1 < \frac{1}{2}, \quad -\frac{1}{2} < y_2 < \frac{1}{2}, \quad Z^- < z < Z^+\right\},$$
$$\text{where } Z^\pm = \pm\frac{1}{2} \pm F^\pm(y)$$
$$\text{and } y = (y_1, y_2), \quad x = (x_1, x_2) \tag{3.2}$$

and the unit normal vector from Eq. 2.3b becomes,

$$n^\pm = \left(\mp\frac{1}{h_1}\frac{\partial F^\pm}{\partial y_1}, \mp\frac{1}{h_2}\frac{\partial F^\pm}{\partial y_2}, 1\right) \times \left[1 + \frac{1}{h_1^2}\left(\frac{\partial F^\pm}{\partial y_1}\right)^2 + \frac{1}{h_2^2}\left(\frac{\partial F^\pm}{\partial y_2}\right)^2\right]^{-1/2} \tag{3.3}$$

Let us now make the following asymptotic assumptions:

$$\begin{aligned} p_\alpha{}^\pm &= \delta^2 r_\alpha(\mathbf{x},\mathbf{y}), \quad p_3^\pm = \delta^3 q_3^\pm(\mathbf{x},\mathbf{y}) \\ P_\alpha &= \delta f_\alpha(\mathbf{x},\mathbf{y},z), \quad P_3 = \delta^2 g_3(\mathbf{x},\mathbf{y},z) \end{aligned} \tag{3.4a}$$

Further, let us assume the following through-the-thickness linear relationships for T, following the commonly adopted assumption in the treatment of heat conduction of plate and shell structures, see for example Podstrigach and Shvets [68], Podstrigach et al. [69].

$$T(\mathbf{x},\mathbf{y},z) = T_1(\mathbf{x},\mathbf{y},z) + zT_2(\mathbf{x},\mathbf{y},z) \tag{3.4b}$$

The reason for the asymptotic forms of Eqs. 2.4b, 2.4c and 3.4a is to ensure convergence of the developed model to its classical plate counterpart as $\delta \to 0$.

The introduction of the fast variables necessitates the transformation of the derivatives according to:

$$\frac{\partial}{\partial x_\alpha} \to \frac{\partial}{\partial x_\alpha} + \frac{1}{\delta h_\alpha}\frac{\partial}{\partial y_\alpha} \quad \text{and} \quad \frac{\partial}{\partial x_3} = \frac{1}{\delta}\frac{\partial}{\partial z} \tag{3.4c}$$

It is noted that the transformations involving x_1 and x_2 have the form shown in Eq. 3.4c in accordance with the two-scale expansion formalism, see e.g. [65], whereas for those involving x_3 we have an ordinary coordinate transformation. One can therefore express the dependent field variables in powers of δ in the form of:

(i) *Basic expansions*

$$u_i(\mathbf{x},\mathbf{y},z) = u_i^{(0)}(\mathbf{x},\mathbf{y},z) + \delta u_i^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2 u_i^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right) \tag{3.5a}$$

$$\varphi(\mathbf{x},\mathbf{y},z) = \delta\left\{\varphi^{(0)}(\mathbf{x},\mathbf{y},z) + \delta\varphi^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2\varphi^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right)\right\} \tag{3.5b}$$

$$\psi(\mathbf{x},\mathbf{y},z) = \delta\left\{\psi^{(0)}(\mathbf{x},\mathbf{y},z) + \delta\psi^{(1)}(\mathbf{x},\mathbf{y},z) \; \delta^2\psi^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right)\right\} \tag{3.5c}$$

$$\begin{aligned} T_1(\mathbf{x},\mathbf{y},z) &= T_1^{(0)}(\mathbf{x},\mathbf{y},z) + \delta T_1^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2 T_1^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right) \\ T_2(\mathbf{x},\mathbf{y},z) &= T_2^{(0)}(\mathbf{x},\mathbf{y},z) + \delta T_2^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2 T_2^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right) \end{aligned} \tag{3.5d}$$

(ii) *Derived expansions*

$$\sigma_{ij}(\mathbf{x},\mathbf{y},z) = \sigma_{ij}^{(0)}(\mathbf{x},\mathbf{y},z) + \delta\sigma_{ij}^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2\sigma_{ij}^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right) \tag{3.5e}$$

$$B_i(\mathbf{x},\mathbf{y},z) = \delta\left\{B_i^{(0)}(\mathbf{x},\mathbf{y},z) + \delta B_i^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2 B_i^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right)\right\} \tag{3.5f}$$

$$D_i(\mathbf{x},\mathbf{y},z) = \delta\left\{D_i^{(0)}(\mathbf{x},\mathbf{y},z) + \delta D_i^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2 D_i^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right)\right\} \tag{3.5g}$$

$$e_{ij}(\mathbf{x},\mathbf{y},z) = e_{ij}^{(0)}(\mathbf{x},\mathbf{y},z) + \delta e_{ij}^{(1)}(\mathbf{x},\mathbf{y},z) + \delta^2 e_{ij}^{(2)}(\mathbf{x},\mathbf{y},z) + O\left(\delta^3\right) \tag{3.5h}$$

Eq. 3.5h can be used in conjunction with Eq. 3.4c and the familiar strain-displacement relationships to obtain the following expressions for the terms of the mechanical strain expansion:

$$\begin{aligned} e_{\alpha\beta}{}^{(m)} &= \frac{1}{2}\left(u_{\alpha,\beta x}{}^{(m)} + u_{\beta,\alpha x}{}^{(m)} + h_\beta^{-1}u_{\alpha,\beta y}{}^{(m+1)} + h_\alpha^{-1}u_{\beta,\alpha y}{}^{(m+1)}\right), \\ e_{3\beta}{}^{(m)} &= \frac{1}{2}\left(u_{3,\beta x}{}^{(m)} + h_\beta^{-1}u_{3,\beta y}{}^{(m+1)} + u_{\beta,z}{}^{(m+1)}\right) \\ e_{33}{}^{(m)} &= u_{3,z}{}^{(m+1)} \qquad m = 0, 1, 2\ldots \end{aligned} \tag{3.6a}$$

Here, and in the sequel, we adopt the following short-hand convention (except in a few instances where the original format is maintained for clarity):

$$\frac{\partial\varphi_\alpha}{\partial y_\beta} = \varphi_{\alpha,\beta y}, \quad \frac{\partial\varphi_\alpha}{\partial x_\beta} = \varphi_{\alpha,\beta x}, \quad \frac{\partial\varphi_\alpha}{\partial z} = \varphi_{\alpha,z} \tag{3.6b}$$

To obtain equivalent expressions for the general terms pertaining to the asymptotic stress field expansion we substitute expressions 3.5a- 3.5e into the constitutive equation 2.4a and compare terms with the same power of δ to obtain:

$$\begin{aligned} \sigma_{ij}{}^{(0)} &= C_{ijk\alpha}\left[u_{k,\alpha x}{}^{(0)} + h_\alpha^{-1}u_{k,\alpha y}{}^{(0)}\right] + C_{ijk3}u_{k,z}{}^{(0)} + e_{\alpha ij}\left[\varphi_{,\alpha x}{}^{(0)} + h_\alpha^{-1}\varphi_{,\alpha y}{}^{(1)}\right] \\ &+ e_{3ij}\varphi_{,z}{}^{(1)} + Q_{\alpha ij}\left[\psi_{,\alpha x}{}^{(0)} + h_\alpha^{-1}\psi_{,\alpha y}{}^{(1)}\right] + Q_{3ij}\psi_{,z}{}^{(1)} \end{aligned}$$

$$\begin{aligned} \sigma_{ij}{}^{(n)} &= C_{ijk\alpha}\left[u_{k,\alpha x}{}^{(n)} + h_\alpha^{-1}u_{k,\alpha y}{}^{(n+1)}\right] + C_{ijk3}u_{k,z}{}^{(n+1)} + e_{\alpha ij}\left[\varphi_{,\alpha x}{}^{(n)} + h_\alpha^{-1}\varphi_{,\alpha y}{}^{(n+1)}\right] \\ &+ e_{3ij}\varphi_{,z}{}^{(n+1)} + Q_{\alpha ij}\left[\psi_{,\alpha x}{}^{(n)} + h_\alpha^{-1}\psi_{,\alpha y}{}^{(n+1)}\right] \\ &+ Q_{3ij}\psi_{,z}{}^{(n+1)} - \Theta_{ij}\left[T_1{}^{(n-1)} + zT_2{}^{(n-1)}\right], \quad n \geq 1 \end{aligned} \tag{3.6c}$$

Similarly, substituting Eqs. 3.5a – 3.5d and 3.5f – 3.5g into the constitutive relations 2.4b and 2.4c, gives the corresponding terms for the electric displacement and magnetic induction:

$$\begin{aligned} D_i{}^{(0)} &= e_{ik\alpha}\left[u_{k,\alpha x}{}^{(0)} + h_\alpha^{-1}u_{k,\alpha y}{}^{(1)}\right] + e_{ik3}u_{k,z}{}^{(1)} - \varepsilon_{i\alpha}\left[\varphi_{,\alpha x}{}^{(0)} + h_\alpha^{-1}\varphi_{,\alpha y}{}^{(1)}\right] \\ &- \varepsilon_{i3}\varphi_{,z}{}^{(1)} - \lambda_{i\alpha}\left[\psi_{,\alpha x}{}^{(0)} + h_\alpha^{-1}\psi_{,\alpha y}{}^{(1)}\right] - \lambda_{i3}\psi_{,\alpha z}{}^{(1)} \end{aligned}$$

$$D_i{}^{(n)} = e_{ik\alpha}\left[u_{k,\alpha x}{}^{(n)} + h_\alpha^{-1}u_{k,\alpha y}{}^{(n+1)}\right] + e_{ik3}u_{k,z}{}^{(n+1)} - \varepsilon_{i\alpha}\varphi_{,\alpha x}{}^{(0)}\left[\varphi_{,\alpha x}{}^{(n)} + h_\alpha^{-1}\varphi_{,\alpha y}{}^{(n+1)}\right]$$

$$- \varepsilon_{i3}\varphi_{,z}{}^{(n+1)} - \lambda_{i\alpha}\left[\psi_{,\alpha x}{}^{(n)} + h_\alpha{}^{-1}\psi_{,\alpha y}{}^{(n+1)}\right] - \lambda_{i3}\psi_{,\alpha z}{}^{(n+1)} + \xi_i\left[T_1{}^{(n-1)} + zT_2{}^{(n-1)}\right], \quad n \geq 1 \tag{3.6d}$$

$$B_i{}^{(0)} = Q_{ik\alpha}\left[u_{k,\alpha x}{}^{(0)} + h_\alpha{}^{-1}u_{k,\alpha y}{}^{(1)}\right] + Q_{ik3}u_{k,z}{}^{(1)} - \lambda_{i\alpha}\left[\varphi_{,\alpha x}{}^{(0)} + h_\alpha{}^{-1}\varphi_{,\alpha y}{}^{(1)}\right] - \lambda_{i3}\varphi_{,z}{}^{(1)} - \mu_{i\alpha}\left[\psi_{,\alpha x}{}^{(0)} + h_\alpha{}^{-1}\psi_{,\alpha y}{}^{(1)}\right] - \mu_{i3}\psi_{,z}{}^{(1)}$$

$$B_i{}^{(n)} = Q_{ik\alpha}\left[u_{k,\alpha x}{}^{(n)} + h_\alpha{}^{-1}u_{k,\alpha y}{}^{(n+1)}\right] + Q_{ik3}u_{k,z}{}^{(n+1)} - \lambda_{i\alpha}\left[\varphi_{,\alpha x}{}^{(n)} + h_\alpha{}^{-1}\varphi_{,\alpha y}{}^{(n+1)}\right] - \lambda_{i3}\varphi_{,z}{}^{(n+1)} - \mu_{i\alpha}\left[\psi_{,\alpha x}{}^{(n)} + h_\alpha{}^{-1}\psi_{,\alpha y}{}^{(n+1)}\right] - \mu_{i3}\psi_{,z}{}^{(n+1)} + \eta_i\left[T_1{}^{(n-1)} + zT_2^{(n-1)}\right], \quad n \geq 1 \tag{3.6e}$$

We note that for reasons of compactness, in Eqs. 3.6a and in the sequel, we will forego the arguments of the functions except when it is deemed necessary to include them for the sake of clarity. It is also important to mention that in the process of deriving expressions 3.6a to 3.6e it is readily discovered that the leading terms in the asymptotic expansions for mechanical displacement and electric and magnetic potentials are independent of the macroscopic variables, y_α and z. Likewise, consideration of a thermal conductivity boundary value problem examined in Kalamkarov [44] and Hadjiloizi et al. [62], leads to the same conclusion for the leading terms of the asymptotic expansion of the temperature field, Eq. 3.5d. Collectively, these observations are summarized as:

$$u_j{}^{(0)} = u_j{}^{(0)}(\mathbf{x}); \quad \varphi^{(0)} = \varphi^{(0)}(\mathbf{x}); \quad \psi^{(0)} = \psi^{(0)}(\mathbf{x}); \quad T_1{}^{(0)} = T_1{}^{(0)}(\mathbf{x}); \quad T_2{}^{(0)} = T_2{}^{(0)}(\mathbf{x})$$

4 Balance laws, boundary conditions and homogenization

Our ultimate objective is to derive the so-called unit cell problems from which the effective coefficients may be extracted. To this end we substitute Eq. 3.5e into Eq. 2.2a and then compare terms with the same power of δ to obtain the following system of differential equations:

$$\begin{aligned} &h_\beta{}^{-1}\sigma_{i\beta,\beta y}{}^{(0)} + \sigma_{i3,z}{}^{(0)} = 0 \\ &\sigma_{i\beta,\beta x}{}^{(0)} + h_\beta{}^{-1}\sigma_{i\beta,\beta y}{}^{(1)} + \sigma_{i3,z}{}^{(1)} = 0 \\ &\sigma_{i\beta,\beta x}{}^{(1)} + h_\beta{}^{-1}\sigma_{i\beta,\beta y}{}^{(2)} + \sigma_{i3,z}{}^{(2)} = f_i \\ &\sigma_{i\beta,\beta x}{}^{(2)} + h_\beta{}^{-1}\sigma_{i\beta,\beta y}{}^{(3)} + \sigma_{i3,z}{}^{(3)} = g_i \quad \text{and} \\ &\sigma_{i\beta,\beta x}{}^{(n)} + h_\beta{}^{-1}\sigma_{i\beta,\beta y}{}^{(n+1)} + \sigma_{i3,z}{}^{(n+1)} = 0, \quad n \geq 3 \end{aligned} \tag{4.1}$$

where we define $f_3 = g_1 = g_2 = 0$. Each of these differential equations must be accompanied by the appropriate boundary condition. To this end, we write Cauchy's expression in 2.3a as,

$$\sigma_{ij}n_j{}^{\pm} = \pm p_i \tag{4.2a}$$

where the negative sign on the right-hand side corresponds to an inward unit normal vector. We then substitute expansion 3.5e into Eq. 4.2a to obtain, in view of Eq. 3.4a, the following boundary conditions to be satisfied at the top and bottom surfaces of the smart composite plate:

$$\begin{aligned} &\sigma_{\alpha j}{}^{(0)}N_j{}^{\pm} + \delta\sigma_{\alpha j}{}^{(1)}N_j{}^{\pm} + \delta^2\sigma_{\alpha j}{}^{(1)}N_j{}^{\pm} + \delta^3\sigma_{\alpha j}{}^{(2)}N_j{}^{\pm} + \cdots = \pm\omega^{\pm}\delta^2 r_\alpha{}^{\pm} \\ &\sigma_{3j}{}^{(0)}N_j{}^{\pm} + \delta\sigma_{3j}{}^{(1)}N_j{}^{\pm} + \delta^2\sigma_{3j}{}^{(1)}N_j{}^{\pm} + \delta^3\sigma_{3j}{}^{(2)}N_j{}^{\pm} + \cdots = \pm\omega^{\pm}\delta^3 q_3{}^{\pm} \end{aligned} \tag{4.2b}$$

Here, for the sake of convenience, the following definitions are made:

$$\mathbf{N}^{\pm} = \left(\mp\frac{1}{h_1}\frac{\partial F^{\pm}}{\partial y_1}, \mp\frac{1}{h_2}\frac{\partial F^{\pm}}{\partial y_2}, 1\right) \quad \text{and} \quad \omega^{\pm} = \sqrt{1 + \frac{1}{h_1{}^2}\left(\frac{\partial F^{\pm}}{\partial y_1}\right)^2 + \frac{1}{h_2{}^2}\left(\frac{\partial F^{\pm}}{\partial y_2}\right)^2} \tag{4.2c}$$

We recall that functions $F^{\pm}$ define the geometric profiles of the top and bottom surfaces of the plate as shown in Fig. 1. Finally, equating like powers of δ gives the final form of the appropriate stress boundary conditions, namely:

$$\left.\begin{aligned} &\sigma_{ij}{}^{(m)}N_j{}^{\pm} = 0, \quad m = 0, 1 \\ &\sigma_{ij}{}^{(2)}N_j{}^{\pm} = \pm\omega^{\pm}r_i{}^{\pm}, \quad r_3{}^{\pm} = 0 \\ &\sigma_{ij}{}^{(3)}N_j{}^{\pm} = \pm\omega^{\pm}q_i{}^{\pm}, \quad q_\alpha{}^{\pm} = 0 \\ &\sigma_{ij}{}^{(n)}N_j{}^{\pm} = 0, \quad n \geq 4 \end{aligned}\right\} \text{ on } Z^{\pm} \tag{4.2d}$$

We proceed in much the same way for the electric displacement problem. Accordingly, keeping Eq. 3.4c in mind, we substitute Eq. 3.5g into the governing Eq. 2.2b and compare terms with the same powers of δ to get the following set of differential equations:

$$\begin{aligned} &h_\beta{}^{-1}D_{\beta,\beta y}{}^{(0)} + D_{3,z}{}^{(0)} = 0 \\ &D_{\beta,\beta x}{}^{(n)} + h_\beta{}^{-1}D_{\beta,\beta y}^{(n+1)} + D_{3,z}{}^{(n+1)} = 0 \quad n \geq 0 \end{aligned} \tag{4.3a}$$

The pertinent boundary conditions are obtained by substituting Eq. 3.5g into the second expression in Eq. 2.3a and comparing terms to obtain:

$$D_i{}^{(n)}N_i{}^{\pm} = 0 \quad \text{for } n \geq 0 \quad \text{on } Z^{\pm} \tag{4.3b}$$

Similar expressions are readily obtained for magnetic induction from Eqs. 2.2c, 3.4c, 3.5f and the third expression in Eq. 2.3a:

$$h_\beta^{-1} B_{\beta,\beta y}{}^{(0)} + B_{3,z}{}^{(0)} = 0$$
$$B_{\beta,\beta x}{}^{(n)} + h_\beta^{-1} B_{\beta,\beta y}{}^{(n+1)} + B_{3,z}{}^{(n+1)} = 0 \quad n \geq 0 \tag{4.4a}$$

and

$$B_i^{(n)} N_i^{\pm} = 0 \quad for\ n \geq 0 \quad on\ Z^{\pm} \tag{4.4b}$$

We next introduce the averaging procedure,

$$\langle \ldots \rangle = \int_\Omega (\ldots)\, dy_1 dy_2 dz \tag{4.5a}$$

defined over the volume $|\Omega|$ of the unit cell Ω with boundary surface $\partial\Omega$, and proceed to show the following relationship,

$$\left\langle h_\alpha^{-1} Q_{\alpha,\alpha y} + Q_{3,z} \right\rangle = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \left(Q_i^+ N_i^+ - Q_i^- N_i^-\right) dy_1 dy_2 \tag{4.5b}$$

where $\mathbf{N}^{\pm}$ is defined in Eq. 4.2c and $Q_i^{\pm}$ are the values Q_i takes on the surfaces $Z^{\pm}$. Starting from the divergence theorem we have

$$\begin{aligned}\left\langle h_\alpha^{-1} Q_{\alpha,\alpha y} + Q_{3,z} \right\rangle &= \int_{\Omega_\delta} \left(h_\alpha^{-1} Q_{\alpha,\alpha y} + Q_{3,z}\right) dv \\ &= \int_{\partial\Omega_\delta} \left(h_\alpha^{-1} Q_\alpha n_{y\alpha} + Q_3 n_{y3}\right) dA,\end{aligned} \tag{4.5c}$$

where $n_y{}^+(n_y{}^-)$ is the outward (inward) unit normal vector defined with respect to the (y_1, y_2, z) coordinate system of the unit cell and is given by:

$$n_y{}^{\pm} = \left(\mp\frac{\partial F^{\pm}}{\partial y_1}, \mp\frac{\partial F^{\pm}}{\partial y_1}, 1\right) \Big/ \sqrt{\left(\frac{\partial F^{\pm}}{\partial y_1}\right)^2 + \left(\frac{\partial F^{\pm}}{\partial y_2}\right)^2 + 1} \tag{4.5d}$$

Now, periodicity considerations stipulate that the first integral in Eq. 4.5c reduces to

$$\int_{S^+} \left(h_\alpha^{-1} Q_\alpha{}^+ n_{y\alpha}{}^+ + Q_3{}^+ n_{y3}{}^+\right) ds_\Omega{}^+ - \int_{S^-} \left(h_\alpha^{-1} Q_\alpha{}^- n_{y\alpha}{}^- + Q_3{}^- n_{y3}{}^-\right) ds_\Omega{}^- \tag{4.5e}$$

where $ds_\Omega{}^+(ds_\Omega{}^-)$ is given by:

$$ds_\Omega{}^{\pm} = \sqrt{\left(\frac{\partial F^{\pm}}{\partial y_1}\right)^2 + \left(\frac{\partial F^{\pm}}{\partial y_2}\right)^2 + 1}\, dy_1 dy_2 \tag{4.5f}$$

Finally, substituting Eq. 4.5f into Eq. 4.5e proves the result in Eq. 4.5b on account of 4.2c.

Within the framework of the terminology adopted in this paper, the force resultants, $N_{i\alpha}$, Q_α, and moment resultants, $M_{i\alpha}$, of the homogenized plate, see Kalamkarov [44], Gibson [70] etc., are given by:

$$\left\langle N_{\alpha\beta} \right\rangle = \delta \left\langle \sigma_{\alpha\beta} \right\rangle, \quad \left\langle Q_\alpha \right\rangle = \delta \left\langle \sigma_{\alpha 3} \right\rangle, \quad \left\langle M_{\alpha\beta} \right\rangle = \delta^2 \left\langle z\sigma_{\alpha\beta} \right\rangle \tag{4.6}$$

To obtain the force resultants we average the expressions in Eq. 4.1 in the sense of Eq. 4.5a and apply at the same time the boundary conditions 4.2d and the general result 4.5b. We get

$$\begin{aligned} &N_{\alpha\beta,x_\beta}{}^{(0)} = 0,\ N_{\alpha\beta,x_\beta}{}^{(1)} + \delta r_\alpha{}^*(x_\gamma) = \delta \langle f_\alpha \rangle, \\ &N_{\alpha\beta,x_\beta}{}^{(n)} = 0 \quad \text{where} \quad n \geq 2 \\ &Q_{\beta,x\beta}{}^{(1)} = 0, \quad Q_{\beta,x\beta}{}^{(2)} + \delta q_3{}^*(x_\gamma) = \delta \langle g_3 \rangle, \\ &Q_{\beta,x\beta}{}^{(n)} = 0 \quad \text{where} \quad n \geq 3 \end{aligned} \tag{4.7a}$$

where we define:

$$r_\alpha{}^*(\mathbf{x}) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \left(\omega^+ r_\alpha{}^+ + \omega^- r_\alpha{}^-\right) dy_1 dy_2$$
$$q_3{}^*(\mathbf{x}) = \int_{1/2}^{1/2}\int_{-1/2}^{1/2} \left(\omega^+ q_3{}^+ + \omega^- q_3{}^-\right) dy_1 dy_2 \tag{4.7b}$$

To obtain the moment resultants we multiply the expressions in Eq. 4.1 by z and then integrate over the volume of the unit cell to obtain, on account of the boundary conditions 4.2d and Eq. 4.5b:

$$\begin{aligned} &M_{\alpha\beta,x\beta}{}^{(0)} + \delta \left\langle Q_\alpha{}^{(1)} \right\rangle = 0, \\ &M_{\alpha\beta,x\beta}{}^{(1)} + \delta^2 \rho_\alpha{}^*(\mathbf{x}) - \delta \left\langle Q_\alpha{}^{(2)} \right\rangle = \delta^2 \langle z f_\alpha \rangle \\ &M_{\alpha\beta,x\beta}{}^{(2)} - \delta \left\langle Q_\alpha{}^{(3)} \right\rangle = 0, \\ &\left\langle z\sigma_{3\beta,x\beta}{}^{(2)} \right\rangle + \sigma_3{}^*(\mathbf{x}) = \langle z g_3 \rangle, \\ &M_{\alpha\beta,x\beta}{}^{(n)} = 0 \quad \text{where} \quad n \geq 3 \end{aligned} \tag{4.7c}$$

Here we define:

$$\rho_\alpha{}^*(\mathbf{x}) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \left(z^+\omega^+ r_\alpha{}^+ + z^-\omega^- r_\alpha{}^-\right) dy_1 dy_2$$
$$\sigma_3{}^*(\mathbf{x}) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \left(z^+\omega^+ q_3{}^+ + z^-\omega^- q_3{}^-\right) dy_1 dy_2 \tag{4.7d}$$

It will later on be seen that as a consequence of the plane stress assumption and the fact that $\sigma_{\alpha\beta}{}^{(0)}$ will turn out to be 0, two important consequences of Eqs. 4.7a and 4.7c are:

$$\left\langle \sigma_{i3}{}^{(1)} \right\rangle = \left\langle z\sigma_{i3}{}^{(1)} \right\rangle = 0 \tag{4.7e}$$

We finally repeat this homogenization procedure on Eqs. 4.3a and 4.4a to give the governing equations for the averaged electric displacement and magnetic induction in the form of:

$$\begin{aligned} &\left\langle D_{\beta,x\beta}{}^{(n)} \right\rangle = 0, \quad n \geq 0 \\ &\left\langle B_{\beta,x\beta}{}^{(n)} \right\rangle = 0, \quad n \geq 0 \end{aligned} \tag{4.7f}$$

5 Unit cell problems for homogenized plate

Our first objective in this section is to determine the leading terms in the asymptotic expansions of mechanical displacement and electrical and magnetic potential. To achieve this, we substitute the first term of the asymptotic expansion (3.5e) given in Eq. (3.6c) for n = 0 into the first expression in (4.1) and the stress boundary condition in (4.2d) for m = 0. After some straightforward *albeit* tedious algebraic manipulations we obtain the following expression and accompanying boundary condition:

$$\begin{aligned}&D_{ik}u_k^{(1)}+C_i\varphi^{(1)}+F_i\psi^{(1)} = -C_{ik\alpha}(\mathbf{y},z)\,u_{k,\alpha x}^{(0)}(\mathbf{x})\\ &-P_{\alpha i}(\mathbf{y},z)\,\varphi_{k,\alpha x}^{(0)} - R_{\alpha i}(\mathbf{y},z)\,\psi_{k,\alpha x}^{(0)}\\ &\left\{L_{ijk}u_k^{(1)}+M_{ij}\varphi^{(1)}+N_{ij}\psi^{(1)}+C_{ijk\alpha}u_{k,\alpha x}^{(0)}(\mathbf{x})\right.\\ &\left.+e_{\alpha ij}\varphi_{k,\alpha x}^{(0)}+Q_{\alpha ij}\psi_{k,\alpha x}^{(0)}\right\}N_j^{\pm}=0 \quad \text{on} \quad Z^{\pm}\end{aligned} \tag{5.1}$$

Here, for economy of notation, we defined the following differential operators

$$\begin{aligned}L_{ijk} &= C_{ijk\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+C_{ijk3}\frac{\partial}{\partial z},\\ M_{ij} &= e_{\alpha ij}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+e_{3ij}\frac{\partial}{\partial z}\\ N_{ij} &= Q_{\alpha ij}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+Q_{3ij}\frac{\partial}{\partial z},\\ D_{ij} &= \frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}L_{i\alpha j}+\frac{\partial}{\partial z}L_{i3j}\\ C_i &= \frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}M_{i\alpha}+\frac{\partial}{\partial z}M_{i3},\\ F_i &= \frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}N_{i\alpha}+\frac{\partial}{\partial z}N_{i3}\end{aligned} \tag{5.2a}$$

and the following parameters

$$\begin{aligned}&C_{ik\alpha}=\frac{1}{h_\beta}\frac{\partial C_{i\beta k\alpha}}{\partial y_\beta}+\frac{\partial C_{i3k\alpha}}{\partial z}, \quad P_{\alpha i}=\frac{1}{h_\beta}\frac{\partial e_{i\alpha\beta}}{\partial y_\beta}+\frac{\partial e_{i\alpha 3}}{\partial z},\\ &R_{\alpha i}=\frac{1}{h_\beta}\frac{\partial Q_{i\alpha\beta}}{\partial y_\beta}+\frac{\partial Q_{i\alpha 3}}{\partial z}\end{aligned} \tag{5.2b}$$

It should be noted that each term on the right-hand side of the first expression of Eq. 5.1 is a product of a function of **x** and a function of **y**, z. This will play a significant role in the general form of the solution of $u_k^{(0)}$, $\varphi^{(0)}$ and $\psi^{(0)}$, as we will see shortly.

Similarly, substitution of Eq. 3.6d for n = 0 into the first expression in Eq. 4.3a and the associated boundary condition in Eq. 4.3b for n = 0 results in the following system:

$$\begin{aligned}&A_j^{*}u_j^{(1)}-L^{*}\varphi^{(1)}-M^{*}\psi^{(1)} = -G_{k\alpha}^{*}(\mathbf{y},z)\,u_{k,\alpha x}^{(0)}(\mathbf{x})\\ &+I_\alpha^{*}(\mathbf{y},z)\,\varphi_{k,\alpha x}^{(0)}(\mathbf{x})+K_\alpha^{*}(\mathbf{y},z)\,\psi_{k,\alpha x}^{(0)}(\mathbf{x})\\ &\left\{L_{ij}^{*}u_k^{(1)}-M_i^{*}\varphi^{(1)}-N_i^{*}\psi^{(1)}+e_{ij\alpha}u_{k,\alpha x}^{(0)}(\mathbf{x})\right.\\ &\left.-\varepsilon_{i\alpha}\varphi_{k,\alpha x}^{(0)}(\mathbf{x})-\lambda_{i\alpha}\psi_{k,\alpha x}^{(0)}(\mathbf{x})\right\}N_j^{\pm}=0 \quad \text{on} \quad Z^{\pm}\end{aligned} \tag{5.3}$$

Here, for the sake of convenience, we define the following differential operators

$$\begin{aligned}&L_{ij}^{*}=e_{ij\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+e_{ij3}\frac{\partial}{\partial z}, \quad M_i^{*}=\varepsilon_{i\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+\varepsilon_{i3}\frac{\partial}{\partial z}\\ &N_i^{*}=\lambda_{i\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+\lambda_{i3}\frac{\partial}{\partial z}, \quad A_i^{*}=\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}L_{\alpha i}^{*}+\frac{\partial}{\partial z}L_{3i}^{*}\\ &L^{*}=\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}M_\alpha^{*}+\frac{\partial}{\partial z}M_3^{*}, \quad M^{*}=\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}N_\alpha^{*}+\frac{\partial}{\partial z}N_3^{*}\end{aligned} \tag{5.4a}$$

and the following parameters

$$\begin{aligned}&G_{k\alpha}^{*}=\frac{1}{h_\beta}\frac{\partial e_{\beta k\alpha}}{\partial y_\beta}+\frac{\partial e_{3k\alpha}}{\partial z}, \quad I_\alpha^{*}=\frac{1}{h_\beta}\frac{\partial \varepsilon_{\alpha\beta}}{\partial y_\beta}+\frac{\partial \varepsilon_{3\beta}}{\partial z},\\ &K_\alpha^{*}=\frac{1}{h_\beta}\frac{\partial \lambda_{\alpha\beta}}{\partial y_\beta}+\frac{\partial \lambda_{3\beta}}{\partial z}\end{aligned} \tag{5.4b}$$

Finally, substitution of Eq. 3.6e for n = 1 into the first expression in Eq. 4.4a and the associated boundary condition in Eq. 4.4b for n = 0 results in the following system:

$$\begin{aligned}&A_ju_j^{(1)}-L\varphi^{(1)}-M\psi^{(1)} = -G_{k\alpha}(\mathbf{y},z)\,u_{k,\alpha x}^{(0)}(\mathbf{x})\\ &+I_\alpha(\mathbf{y},z)\,\varphi_{k,\alpha x}^{(0)}(\mathbf{x})+K_\alpha(\mathbf{y},z)\,\psi_{k,\alpha x}^{(0)}(\mathbf{x})\\ &\left\{L_{ij}u_k^{(1)}-M_i\varphi^{(1)}-N_i\psi^{(1)}+q_{ij\alpha}u_{k,\alpha x}^{(0)}(\mathbf{x})\right.\\ &\left.-\lambda_{i\alpha}\varphi_{k,\alpha x}^{(0)}(\mathbf{x})-\mu_{i\alpha}\psi_{k,\alpha x}^{(0)}(\mathbf{x})\right\}N_j^{\pm}=0 \quad \text{on} \quad Z^{\pm}\end{aligned} \tag{5.5}$$

Here we define the following differential operators

$$\begin{aligned}&L_{ij}=q_{ij\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+q_{ij3}\frac{\partial}{\partial z}, \quad M_i=\lambda_{i\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+\lambda_{i3}\frac{\partial}{\partial z}\\ &N_i=\mu_{i\alpha}\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}+\mu_{i3}\frac{\partial}{\partial z}, \quad A_i=\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}L_{\alpha i}+\frac{\partial}{\partial z}L_{3i}\\ &L=\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}M_\alpha+\frac{\partial}{\partial z}M_3, \quad M=\frac{1}{h_\alpha}\frac{\partial}{\partial y_\alpha}N_\alpha+\frac{\partial}{\partial z}N_3\end{aligned} \tag{5.6a}$$

and the following parameters

$$\begin{aligned}&G_{k\alpha}=\frac{1}{h_\beta}\frac{\partial q_{\beta k\alpha}}{\partial y_\beta}+\frac{\partial q_{3k\alpha}}{\partial z}, \quad I_\alpha=\frac{1}{h_\beta}\frac{\partial \lambda_{\alpha\beta}}{\partial y_\beta}+\frac{\partial \lambda_{3\beta}}{\partial z}=K_\alpha^{*},\\ &K_\alpha=\frac{1}{h_\beta}\frac{\partial \mu_{\alpha\beta}}{\partial y_\beta}+\frac{\partial \mu_{3\beta}}{\partial z}\end{aligned} \tag{5.6b}$$

The separation of variables on the right-hand sides of the differential equations in (5.1), (5.3) and (5.5) allows us to write down the solution of $u_k^{(0)}$, $\varphi^{(0)}$ and $\psi^{(0)}$ in the form of:

$$u_i^{(1)}(\mathbf{x},\mathbf{y},z) = N_i^{k\alpha}(\mathbf{y},z)\,u_{k,\alpha x}^{(0)}(\mathbf{x}) + M_\alpha^{\,i}(\mathbf{y},z)\,\varphi_{k,\alpha x}^{(0)}(\mathbf{x}) + N_\alpha^{\,i}(\mathbf{y},z)\,\psi_{k,\alpha x}^{(0)}(\mathbf{x}) + \omega_i(\mathbf{x}) \tag{5.7a}$$

$$\varphi^{(1)}(\mathbf{x},\mathbf{y},z) = A_{k\alpha}(\mathbf{y},z)\,u_{k,\alpha x}^{(0)}(\mathbf{x}) + \Xi_\alpha(\mathbf{y},z)\,\varphi_{k,\alpha x}^{(0)}(\mathbf{x}) + O_\alpha(\mathbf{y},z)\,\psi_{k,\alpha x}^{(0)}(\mathbf{x}) + \gamma(\mathbf{x}) \tag{5.7b}$$

$$\psi^{(1)}(\mathbf{x},\mathbf{y},z) = \Lambda_{k\alpha}(\mathbf{y},z)\,u_{k,\alpha x}^{(0)}(\mathbf{x}) + Z_\alpha(\mathbf{y},z)\,\varphi_{k,\alpha x}^{(0)}(\mathbf{x}) + \Gamma_\alpha(\mathbf{y},z)\,\psi_{k,\alpha x}^{(0)}(\mathbf{x}) + \omega(\mathbf{x}) \tag{5.7c}$$

Excluding functions $\omega_i(\mathbf{x})$, $\gamma(\mathbf{x})$, $\omega(\mathbf{x})$ which are the homogeneous solutions Eqs. 5.7a, 5.7b and 5.7c, contain 9 unknown functions, $N_i^{k\alpha}$, M_i^{α}, N_i^{α}, $A_{k\alpha}$, Ξ_α, O_α, $\Lambda_{k\alpha}$, Z_α, Γ_α which are solved by back-substitution into Eqs. 5.1, 5.3 and 5.5 to generate the unit cell problems. These are:

$$h_\beta^{-1} b_{i\beta,\beta y}^{k\alpha}(\mathbf{y},z) + b_{i3,z}^{\;k\alpha}(\mathbf{y},z) = 0 \quad \text{with} \quad b_{ij}^{\;k\alpha}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.8a}$$

$$h_\beta^{-1} b_{\alpha,y}^{\;i\beta}(\mathbf{y},z) + b_{\alpha,z}^{\;i3}(\mathbf{y},z) = 0 \quad \text{with} \quad b_\alpha^{\;ij}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.8b}$$

$$h_\beta^{-1} a_{\alpha,\beta y}^{\;i\beta}(\mathbf{y},z) + a_{\alpha,z}^{\;i3}(\mathbf{y},z) = 0 \quad \text{with} \quad a_\alpha^{\;ij}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.8c}$$

$$h_\beta^{-1} \eta_{\beta,\beta y}^{i\alpha}(\mathbf{y},z) + \eta_{3,z}^{\;i\alpha}(\mathbf{y},z) = 0 \quad \text{with} \quad \eta_j^{\;i\alpha}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.9a}$$

$$h_\beta^{-1} a_{\beta\alpha,\beta y}(\mathbf{y},z) + a_{3\alpha,z}(\mathbf{y},z) = 0 \quad with \quad a_{j\alpha}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.9b}$$

$$h_\beta^{-1} \gamma_{\beta\alpha,\beta\alpha}(y,z) + \gamma_{3\alpha,z}(y,z) = 0 \quad \text{with} \quad \gamma_{j\alpha}(y,z) N_j^{\pm} = 0 \quad on \quad Z_{pm} \tag{5.9c}$$

$$h_\beta^{-1} \delta_{\beta,\beta y}^{\;i\alpha}(\mathbf{y},z) + \delta_{3,z}^{\;i\alpha}(\mathbf{y},z) = 0 \quad \text{with} \quad \delta_j^{\;i\alpha}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.10a}$$

$$h_\beta^{-1} \delta_{\beta\alpha,\beta y}(\mathbf{y},z) + \delta_{3\alpha,z}(\mathbf{y},z) = 0 \quad \text{with} \quad \delta_{j\alpha}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.10b}$$

$$h_\beta^{-1} \xi_{\beta\alpha,\beta y}(\mathbf{y},z) + \xi_{3\alpha,z}(\mathbf{y},z) = 0 \quad \text{with} \quad \xi_{j\alpha}(\mathbf{y},z)\,N_j^{\pm} = 0 \quad on\ Z^{\pm} \tag{5.10c}$$

The unit cell problems can be viewed as being grouped in three separate sets. The first set of three unit cell problems, Eqs. 5.8a – 5.8c, pertains to the mechanical stress problem (force balance equation), the equations of the second set, Eqs. 5.9a – 5.9c, stem from Maxwell's Law for the magnetic field, and the third set of three equations, Eqs. 5.10a – 5.10c, are related to Maxwell's Law involving electric displacement. We also note that unlike unit cell problems in "classical" homogenization schemes, see for example Bakhvalov and Panasenko [42], the unit cell problems in Eqs. 5.8a- 5.10c also involve boundary conditions on the upper and lower surfaces of the unit cell. The following definitions are used in the aforementioned equations.

$$b_{ij}^{\;k\alpha} = L_{ijm} N_m^{\;k\alpha} + M_{ij} A_{k\alpha} + N_{ij} \Lambda_{k\alpha} + C_{ijk\alpha} \tag{5.11a}$$

$$b_\alpha^{\;ij} = L_{ijm} M_\alpha^{\;m} + M_{ij} \Xi_\alpha + N_{ij} Z_\alpha + e_{\alpha ij} \tag{5.11b}$$

$$a_\alpha^{\;ij} = L_{ijm} N_\alpha^{\;m} + M_{ij} O_\alpha + N_{ij} \Gamma_\alpha + Q_{\alpha ij} \tag{5.11c}$$

$$\eta_j^{\;k\alpha} = L_{ji} N_i^{\;k\alpha} - M_j A_{k\alpha} - N_j(y)\,\Lambda_{k\alpha} + Q_{jk\alpha} \tag{5.12a}$$

$$a_{j\alpha} = L_{ji} M_\alpha^{\;i} - M_j \Xi_\alpha - N_j Z_\alpha - \lambda_{j\alpha} \tag{5.12b}$$

$$\gamma_{j\alpha} = L_{ji} N_\alpha^{\;i} - M_j O_\alpha - N_j \Gamma_\alpha - \mu_{j\alpha} \tag{5.12c}$$

$$\delta_j^{\;k\alpha} = L_{ji}^{\;*} N_i^{\;k\alpha} - M_j^{\;*} A_{k\alpha} - N_j^{\;*} \Lambda_{k\alpha} + e_{jk\alpha} \tag{5.13a}$$

$$\delta_{j\alpha} = L_{ji}^{\;*} M_\alpha^{\;i} - M_j^{\;*} \Xi_\alpha - N_j^{\;*} Z_\alpha - \varepsilon_{j\alpha} \tag{5.13b}$$

$$\xi_{j\alpha} = L_{ji}^{\;*} N_\alpha^{\;i} - M_j^{\;*} O_\alpha - N_j^{\;*} \Gamma_\alpha - \lambda_{j\alpha} \tag{5.13c}$$

As their name suggests, the unit cell problems are solved entirely on the domain of the unit cell and are entirely independent of the macroscopic variable. As we will see later on, nine more unit cell problems will be generated six of which will relate to the out-of-plane deformation of the homogenized plate.

In the analysis of the homogenized plate model the possibility of finding an exact solution often plays a significant role. In our case an exact solution for k, α = 3, 1 and 3, 2 can be readily found from Eqs. 5.8a, 5.9a and 5.10a and is of the form:

$$\begin{aligned} &N_1^{\;31} = -z; \quad N_2^{\;31} = 0; \quad N_1^{\;32} = 0; \\ &N_2^{\;32} = -z; \quad N_3^{\;31} = 0; \quad N_3^{\;32} = 0; \\ &\Lambda_{31} = \Lambda_{32} = 0 \\ &A_{31} = A_{32} = 0; \end{aligned} \tag{5.14a}$$

With these results in mind it is readily shown that:

$$b_{ij}^{\;31} = b_{ij}^{\;32} = \eta_j^{\;31} = \eta_j^{\;32} = \delta_j^{\;31} = \delta_j^{\;32} = 0 \tag{5.14b}$$

Since the "3" superscript in the local functions $b_{ij}^{\;kl}$, $\eta_j^{\;kl}$, and $\delta_j^{\;kl}$ becomes obsolete, only $b_{ij}^{\;\alpha l}$, $\eta_j^{\;\alpha l}$, and $\delta_j^{\;\alpha l}$ need to be considered. Consequently, the 1st, 4th and 7th unit cell problems, Eqs. 5.8a, 5.9a and 5.10a may be simplified as follows:

$$h_\beta^{-1} b^{\mu\alpha}_{i\beta,\beta y}(\mathbf{y},z) + b^{\mu\alpha}_{i3,z}(\mathbf{y},z) = 0 \quad \text{with} \quad b^{\mu\alpha}_{i\beta}(\mathbf{y},z) N_j^{\pm} = 0 \quad \text{on } Z^{\pm} \tag{5.15a}$$

$$h_\beta^{-1} \eta^{\mu\alpha}_{\beta,\beta y}(\mathbf{y},z) + \eta^{\mu\alpha}_{3,z}(\mathbf{y},z) = 0 \quad \text{with} \quad b^{ij}_{\alpha}(\mathbf{y},z) N_j^{\pm} = 0 \quad \text{on } Z^{\pm} \tag{5.15b}$$

$$h_\beta^{-1} \delta^{\mu\alpha}_{\beta,\beta y}(y,z) + \delta^{\mu\alpha}_{3,z}(y,z) = 0 \quad \text{with} \quad \delta_j^{\mu\alpha}(y,z) N_j^{\pm} = 0 \quad \text{on } Z^{\pm} \tag{5.15c}$$

Furthermore, expressions 5.7a – 5.7c are simplified as follows:

$$\begin{aligned} u_1^{(1)} &= -z u_{3,1x}^{(0)} + N_1^{\beta\alpha} u_{\beta,\alpha x}^{(0)} + M_\alpha^{1} \varphi_{,\alpha x}^{(0)} + N_\alpha^{1} \psi_{,\alpha x}^{(0)} + \omega_1 \\ u_2^{(1)} &= -z u_{3,2x}^{(0)} + N_2^{\beta\alpha} u_{\beta,\alpha x}^{(0)} + M_\alpha^{2} \varphi_{,\alpha x}^{(0)} + N_\alpha^{2} \psi_{,\alpha x}^{(0)} + \omega_2 \\ u_3^{(1)} &= N_3^{\beta\alpha} u_{\beta,\alpha x}^{(0)} + M_\alpha^{3} \varphi_{,\alpha x}^{(0)} + N_\alpha^{3} \psi_{,\alpha x}^{(0)} + \omega_3 \end{aligned} \tag{5.16a}$$

$$\varphi^{(1)} = A_{\beta\alpha} u_{\beta,\alpha x}^{(0)} + \Xi_\alpha \varphi_{,\alpha x}^{(0)} + O_\alpha \psi_{,\alpha x}^{(0)} + \gamma \tag{5.16b}$$

$$\psi^{(1)} = \Lambda_{\beta\alpha} u_{\beta,\alpha x}^{(0)} + Z_\alpha \varphi_{,\alpha x}^{(0)} + \Gamma_\alpha \psi_{,\alpha x}^{(0)} + \omega \tag{5.16c}$$

Further, an examination of the thermal conductivity boundary value problem and associated Fourier's Law of conduction, see Kalamkarov [44], reveals that second terms of the asymptotic expansion of the temperature field are given by

$$T_1^{(1)} = \beta_\alpha T_{1,\alpha x}^{(0)} + \lambda_1^{(1)}, \quad z T_2^{(1)} = \beta_\alpha^{(1)} T_{2,\alpha x}^{(0)} + \lambda_2^{(1)} \tag{5.17}$$

where β_α, $\beta_\alpha^{(1)}$ are local functions similar to, for example, $A_{\beta\alpha}$ in Eq. 5.16b, and $\lambda_1^{(1)}$, $\lambda_2^{(1)}$ are the solutions of the homogeneous thermal conductivity boundary value problems, see Kalamkarov [44], similar to the γ or ω functions in Eqs. 5.16b and 5.16c.

We are now in a position to compute the leading terms in the expansions of mechanical stress, electric displacement, magnetic induction and mechanical strain. We recall that these variables were defined in Eqs. 3.5e – 3.5h as the "derived" variables. To this end we substitute the results of Eqs. 5.7a – 5.7c into expression 3.6c for n = 0, and keeping Eq. 5.14b in mind, we obtain for $\sigma_{ij}^{(0)}$:

$$\sigma_{ij}^{(0)} = b_{ij}^{\alpha\beta} u_{\alpha,\beta x}^{(0)} + b_\alpha^{ij} \varphi_{,\beta x}^{(0)} + a_\alpha^{ij} \psi_{,\beta x}^{(0)} \tag{5.18a}$$

Here, definitions 5.11a – 5.11c were used. In a similar manner, we substitute Eqs. 5.7a – 5.7c into the first expressions in Eqs. 3.6d and 3.6e to obtain, in view of definitions 5.12a – 5.12c, 5.13a – 5.13c and the results in Eq. 5.14b, the leading terms in the asymptotic expansions for the electric displacement and magnetic induction. They are:

$$D_i^{(0)} = \delta_i^{\alpha\beta} u_{\alpha,\beta x}^{(0)} + \delta_{i\beta} \varphi_{,\beta x}^{(0)} + \xi_{i\beta} \psi_{,\beta x}^{(0)} \tag{5.18b}$$

$$B_i^{(0)} = \eta_i^{\alpha\beta} u_{\alpha,\beta x}^{(0)} + a_{i\beta} \varphi_{,\beta x}^{(0)} + \gamma_{i\beta} \psi_{,\beta x}^{(0)} \tag{5.18c}$$

Finally, the mechanical strain is computed from Eq. 3.6a:

$$\begin{aligned} 2e_{\alpha\beta}^{(0)} &= Q_{\mu\nu}^{\alpha\beta} u_{\alpha,\beta x}^{(0)} + I_{\alpha\beta}^{\mu} \varphi_{,\beta\mu}^{(0)} + S_{\alpha\beta}^{\mu} \psi_{,\beta\mu}^{(0)} \\ 2e_{3\beta}^{(0)} &= \tilde{Q}_{\mu\alpha\beta} u_{\alpha,\beta x}^{(0)} + \tilde{I}_{\beta\mu} \varphi_{,\beta\mu}^{(0)} + \tilde{S}_{\beta\mu} \psi_{,\beta\mu}^{(0)} \\ e_{33}^{(0)} &= N_{3,z}^{\alpha\beta} u_{\alpha,\beta x}^{(0)} + M_{\mu,z}^{3} \varphi_{,\beta\mu}^{(0)} + N_{\mu,z}^{3} \psi_{,\beta\mu}^{(0)} \end{aligned} \tag{5.18d}$$

wherein the following definitions are used:

$$\begin{aligned} Q_{\mu\nu}^{\alpha\beta} &= \left[h_\nu^{-1} N_{\mu,\mu y}^{\alpha\beta} + h_\mu^{-1} N_{\nu,\mu y}^{\alpha\beta} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu} \right] \\ I_{\alpha\beta}^{\mu} &= \left[h_\beta^{-1} M_{\mu,\beta y}^{\alpha} + h_\alpha^{-1} M_{\mu,\alpha y}^{\beta} \right], \\ S_{\alpha\beta}^{\mu} &= \left[h_\beta^{-1} N_{\mu,\beta y}^{\alpha} + h_\alpha^{-1} N_{\mu,\alpha y}^{\beta} \right] \end{aligned} \tag{5.19a}$$

$$\begin{aligned} \tilde{Q}_{\mu\alpha\beta} &= \left[h_\beta^{-1} N_{3,\beta y}^{\mu\alpha} + N_{\beta,z}^{\mu\alpha} \right], \\ \tilde{I}_{\beta\mu} &= \left[h_\beta^{-1} M_{\alpha,\beta y}^{3} + M_{\alpha,z}^{\beta} \right], \\ \tilde{S}_{\beta\mu} &= \left[h_\beta^{-1} N_{\alpha,\beta y}^{3} + N_{\alpha,z}^{\beta} \right] \end{aligned} \tag{5.19b}$$

Averaging Eqs. 5.18a- 5.18c in the sense of Eq. 4.5a gives the pertinent homogenized expressions:

$$\left\langle \sigma_{ij}^{(0)} \right\rangle = \left\langle b_{ij}^{\alpha\beta} \right\rangle u_{\alpha,\beta x}^{(0)} + \left\langle b_\alpha^{ij} \right\rangle \varphi_{,\alpha x}^{(0)} + \left\langle a_\alpha^{ij} \right\rangle \psi_{,\alpha x}^{(0)} \tag{5.20a}$$

$$\left\langle D_i^{(0)} \right\rangle = \left\langle \delta_i^{\alpha\beta} \right\rangle u_{\alpha,\beta x}^{(0)} + \left\langle \delta_{i\alpha} \right\rangle \varphi_{,\alpha x}^{(0)} + \left\langle \xi_{i\alpha} \right\rangle \psi_{,\alpha x}^{(0)} \tag{5.20b}$$

$$\left\langle B_i^{(0)} \right\rangle = \left\langle \hat{a}_i^{\alpha\beta} \right\rangle u_{\alpha,\beta x}^{(0)} + \left\langle a_{i\alpha} \right\rangle \varphi_{,\alpha x}^{(0)} + \left\langle \gamma_{i\alpha} \right\rangle \psi_{,\alpha x}^{(0)} \tag{5.20c}$$

If we subsequently take the expansions in 3.5a- 3.5c and substitute them into the boundary conditions on the lateral surfaces of the thin plate in Eq. 2.3c, we arrive at the following expressions after comparing terms with like powers of δ:

$$\left. \begin{aligned} & u_i^{(n)} = 0, \quad n \geq 0 \\ & \varphi^{(0)} = 0, \quad \varphi^{(1)} = e, \quad \varphi^{(n)} = 0, \quad n \geq 2 \\ & \psi^{(0)} = 0, \quad \psi^{(1)} = h, \quad \psi^{(n)} = 0, \quad n \geq 2 \end{aligned} \right\} \text{ on the lateral surfaces} \tag{5.21}$$

If we substitute the homogenized fields given in Eqs. 5.20a – 5.20c into both the first expression of Eq. 4.7a and the governing equations in 4.7f for n = 0 we readily see that the solution of the resulting differential equations, in conjunction with the boundary conditions in Eq. 5.21, is:

$$u_\alpha{}^{(0)} = 0; \qquad \varphi^{(0)} = 0\,; \qquad \psi^{(0)} = 0 \tag{5.22}$$

It then follows from Eqs. (5.18a)-(5.18d) that:

$$\sigma_{ij}{}^{(0)} = 0\,; \qquad D_i{}^{(0)} = 0\,; \qquad B_i{}^{(0)} = 0\,; \qquad e_{ij}{}^{(0)} = 0 \tag{5.23}$$

With these results in mind, the expressions in Eqs. 5.16a-5.16c simplify to:

$$\begin{aligned} u_1{}^{(1)} &= -zu_{3,1x}{}^{(0)} + \omega_1(x) \\ u_2{}^{(1)} &= -zu_{3,2x}{}^{(0)} + \omega_2(x) \\ u_3{}^{(1)} &= \omega_3(x) \end{aligned} \tag{5.24a}$$

$$\varphi^{(1)} = \gamma(x) \tag{5.24b}$$

$$\psi^{(1)} = \omega(x) \tag{5.24c}$$

In view of the results in Eq. 5.23, the leading terms in the asymptotic expansions for mechanical stress, electric displacement, magnetic induction and mechanical strain are, respectively, $\sigma_{ij}{}^{(1)}$, $D_i{}^{(1)}$, $B_i{}^{(1)}$ and $e_{ij}{}^{(1)}$. We now proceed to calculate them. To this end, we substitute Eqs. 5.24a – 5.24c into Eq. 3.6c for n = 1 to obtain:

$$\begin{aligned} \sigma_{ij}{}^{(1)} = \; & C_{ij\alpha\beta}\left\{-zu_{3,\alpha\beta x}{}^{(0)} + \omega_{\alpha,\beta x}\right\} + C_{ij3\beta}\omega_{3,\beta x} + \\ & + C_{ijk\beta}h_\beta{}^{-1}u_{k,\beta y}{}^{(2)} + C_{ijk3}u_{k,z}{}^{(2)} \\ & + e_{\alpha ij}\left\{\gamma_{,\alpha x} + h_\alpha{}^{-1}\varphi_{\alpha y}{}^{(2)}\right\} + e_{3ij}\varphi_z{}^{(2)} + \\ & + Q_{\alpha ij}\left\{\omega_{,\alpha x} + h_\alpha{}^{-1}\psi_{\alpha y}{}^{(2)}\right\} + Q_{3ij}\psi_z{}^{(2)} + \\ & - \Theta_{ij}\left\{T_1{}^{(0)} + zT_2{}^{(0)}\right\} \end{aligned} \tag{5.25a}$$

Similarly, Eqs. 5.24a- 5.24c are substituted into Eqs. 3.6d and 3.6e for n = 0 to obtain the corresponding expressions for the electric displacement and magnetic induction. They are of the form

$$\begin{aligned} D_i{}^{(1)} = \; & e_{i\alpha\beta}\left\{-zu_{3,\alpha\beta x}{}^{(0)} + \omega_{\alpha,\beta x}\right\} + e_{i3\beta}\omega_{3,\beta x} + \\ & + e_{ij\beta}h_\beta{}^{-1}\partial u_{j,\beta y}{}^{(2)} + e_{ij3}u_{j,z}{}^{(2)} + \\ & - \varepsilon_{i\alpha}\left\{\gamma_{,\alpha x} + h_\alpha{}^{-1}\varphi_{,\alpha y}{}^{(2)}\right\} - \varepsilon_{i3}\varphi_z{}^{(2)} + \\ & - \lambda_{i\alpha}\left\{\omega_{,\alpha x} + h_\alpha{}^{-1}\psi_{,\alpha y}{}^{(2)}\right\} - \lambda_{i3}\psi_{,z}{}^{(2)} + \\ & + \xi_i\left\{T_1{}^{(0)} + zT_2{}^{(0)}\right\} \end{aligned} \tag{5.25b}$$

$$\begin{aligned} B_i{}^{(1)} = \; & Q_{i\alpha\beta}\left\{-zu_{3,\alpha\beta x}{}^{(0)} + \omega_{\alpha,\beta x}\right\} + Q_{i3\beta}\omega_{3,\beta x} + \\ & + Q_{ij\beta}h_\beta{}^{-1}u_{j,\beta y}{}^{(2)} + Q_{ij3}u_{j,z}{}^{(2)} + \\ & - \lambda_{i\alpha}\left\{\gamma_{,\alpha x} + h_\alpha{}^{-1}\varphi_{,\alpha y}{}^{(2)}\right\} - \lambda_{i3}\varphi_{,z}{}^{(2)} + \\ & - \mu_{i\alpha}\left\{\omega_{,\alpha x} + h_\alpha{}^{-1}\psi_{,\alpha y}{}^{(2)}\right\} - \mu_{i3}\psi_{,z}{}^{(2)} + \\ & + \eta_i\left\{T_1{}^{(0)} + zT_2{}^{(0)}\right\} \end{aligned} \tag{5.25c}$$

Finally, the mechanical strain is obtained from Eqs. 3.6a and 5.24a:

$$\begin{aligned} e_{\alpha\alpha}{}^{(1)} &= -zu_{3,\alpha\alpha x}{}^{(0)} + \omega_{\alpha,\alpha x} + \\ & \quad + h_\alpha^{-1}u_{\alpha,\alpha y}{}^{(2)}, \quad \text{no summation on } \alpha \\ 2e_{12}{}^{(1)} &= -2zu_{3,x_1x_2}{}^{(0)} + \omega_{1,x_2} + \omega_{2,x_1} + h_1{}^{-1}u_{2,y_1}{}^{(2)} + \\ & \quad + h_2{}^{-1}u_{1,y_2}{}^{(2)} \\ 2e_{3\beta}{}^{(1)} &= \omega_{3,\beta x} + u_{\beta,z}{}^{(2)} + h_\beta{}^{-1}u_{3,\beta y}{}^{(2)}, \\ e_{33}{}^{(1)} &= u_{3,z}{}^{(2)} \end{aligned} \tag{5.25d}$$

The next step is to solve for the next terms in the asymptotic expansions for the mechanical displacement and the two potential functions. We begin by substituting Eq. 5.25a into the second expression in Eq. 4.1 and into the boundary condition in Eq. 4.2d for m = 1. In view of Eq. 5.23 and the definitions in Eqs. 5.2a and 5.2b, we arrive at the following expressions:

$$\begin{aligned} & D_{ij}u_j{}^{(2)} + C_i\varphi^{(2)} + F_i\psi^{(2)} = -C_{ij\alpha}\omega_{j,\alpha x} - P_{\alpha i}\gamma_{,\alpha x} - R_{\alpha i}\omega_{,\alpha x} \\ & + U_iT_1{}^{(0)} + (\Theta_{i3} + zU_i)\,T_2{}^{(0)} + \left(C_{i3\alpha\beta} + zC_{i\alpha\beta}\right)u^{(0)}_{3,x_\alpha x_\beta}\left\{L_{ijk}u_k{}^{(2)}\right. \\ & + M_{ij}\varphi^{(2)} + N_{ij}\psi^{(2)} + C_{ijk\alpha}\omega_{k,\alpha x} + e_{\alpha ij}\gamma_{,\alpha x} + Q_{\alpha ij}\omega_{,\alpha x} \\ & \left. - \Theta_{ij}\left[T_1{}^{(0)} + zT_2{}^{(0)}\right] - zC_{ij\alpha\beta}u^{(0)}_{3,x_\alpha x_\beta}\right\}N_j^{\pm} = 0 \quad \text{on} \quad Z^{\pm} \end{aligned} \tag{5.26a}$$

Here, we also make the following definition:

$$U_i = h_\beta{}^{-1}\Theta_{i\beta,y_\beta} + \Theta_{i3,z} \tag{5.26b}$$

We repeat this procedure by substituting Eq. 5.25c into the second expression in Eq. 4.4a and the boundary condition in Eq. 4.4b for n = 1. Keeping the results in Eq. 5.23 and the definitions in Eqs. 5.6a and 5.6b in mind, we arrive at the following expressions:

$$\begin{aligned} & A_ju_j{}^{(2)} - L\varphi^{(2)} - M\psi^{(2)} = -G_{j\alpha}\omega_{j,x_\alpha} + I_\alpha\gamma_{,x_\alpha} + K_\alpha\omega_{,x_\alpha} \\ & - VT_1{}^{(0)} + -(\eta_3 + zV)\,T_2{}^{(0)} + \left(Q_{3\alpha\beta} + zG_{\alpha\beta}\right)u^{(0)}_{3,x_\alpha x_\beta}\left\{L_{ij}u_j{}^{(2)}\right. \\ & - M_i\varphi^{(2)} - N_i\psi^{(2)} + Q_{ij\alpha}\omega_{j,x_\alpha} - \lambda_{i\alpha}\gamma_{,x_\alpha} + -\mu_{i\alpha}\omega_{,x_\alpha} \\ & \left. + \eta_i\left[T_1{}^{(0)} + zT_2{}^{(0)}\right] - zQ_{i\alpha\beta}u_{3,x_\alpha x_\beta}{}^{(0)}\right\}N_i^{\pm} = 0 \quad \text{on} \quad Z^{\pm} \end{aligned} \tag{5.26c}$$

Here, we use the following definition:

$$V = \frac{1}{h_\beta}\frac{\partial\eta_\beta}{\partial y_\beta} + \frac{\partial\eta_3}{\partial z} \tag{5.26d}$$

The last differential equation and its associated boundary condition are determined in the same way, by substituting Eq. 5.25b into the second expression in Eq. 4.3a and into the boundary condition in Eq. 4.3b for n = 1. Recalling the results in Eqs. 5.4a, 5.4b and 5.23 we obtain

$$A_j{}^*u_j{}^{(2)} - L^*\varphi^{(2)} - M^*\psi^{(2)} = -G_{j\alpha}{}^*\omega_{j,x_\alpha} + I_\alpha{}^*\gamma_{,x_\alpha} + K_\alpha{}^*\omega_{,x_\alpha}$$

$$- V^{\star} T_1{}^{(0)} + - \left(\xi_3 + zV^{\star}\right) T_2{}^{(0)} + \left(e_{3\alpha\beta} + zG_{\alpha\beta}{}^{\star}\right) u^{(0)}_{3,x_\alpha \partial x_\beta}$$
$$\times \left\{ L_{ij}{}^{\star} u_j{}^{(2)} - M_i{}^{\star} \varphi^{(2)} - N_i{}^{\star} \psi^{(2)} + e_{ij\alpha}\omega_{j,x_\alpha} - \epsilon_{i\alpha}\gamma_{,x_\alpha} \right.$$
$$\left. -\lambda_{i\alpha}\omega_{,x_\alpha} + \zeta_i [T_1{}^{(0)} + zT_2{}^{(0)}] - ze_{i\alpha\beta} u_{3,x_\alpha \partial x_\beta}{}^{(0)} \right\} N_j{}^{\pm} = 0 \ \text{ on } Z^{\pm} \quad (5.26e)$$

where we use the following definition:

$$V^{\star} = \frac{1}{h_\beta}\frac{\partial \xi_\beta}{\partial y_\beta} + \frac{\partial \xi_3}{\partial z} \quad (5.26f)$$

Again, we observe that all terms on the right hand side of the differential equations in 5.26a, 5.26c and 5.26e are products of a function of **x** and a function of **y**. This separation of variables suggests that we can write down the solution of $u_k{}^{(2)}(\mathbf{x},\mathbf{y},z)$, $\varphi^{(2)}(\mathbf{x},\mathbf{y},z)$ and $\psi^{(2)}(\mathbf{x},\mathbf{y},z)$ in the form of the following linear combinations:

$$\begin{aligned} u_i{}^{(2)} = \ & N_i^{j\beta}\omega_{j,x_\beta} + M_\alpha{}^i \gamma_{,x_\beta} + N_\alpha{}^i \omega_{,x_\alpha} + G_i T_1{}^{(0)} + \\ & + G_i{}^{(1)} T_2{}^{(0)} - N_i{}^{(1)\alpha\beta} \hat{u}^{(0)}_{3,x_\alpha x_\beta} + \omega_i{}^{(\star)}(\mathbf{x}) \end{aligned} \quad (5.27a)$$

$$\begin{aligned} \varphi^{(2)} = \ & A_{k\alpha}\omega_{k,x_\alpha} + \Xi_\alpha \gamma_{,x_\alpha} \frac{\partial}{\partial x_\alpha} + O_\alpha \omega_{,x_\alpha} + \Pi T_1{}^{(0)} + \\ & + \Pi^{(1)} T_2{}^{(0)} - A_{\alpha\beta}{}^{(1)} \hat{u}^{(0)}_{3,x_\alpha x_\beta} + \gamma^{\star}(x) \end{aligned} \quad (5.27b)$$

$$\begin{aligned} \psi^{(2)} = \ & \Lambda_{k\alpha}\omega_{k,x_\alpha} + Z_\alpha \gamma_{,x_\alpha} + \Gamma_\alpha \omega_{,x_\alpha} + \Delta T_1^{(0)} + \\ & + \Delta^{(1)} T_2{}^{(0)} - \Lambda^{(1)}_{\alpha\beta} \hat{u}^{(0)}_{3,x_\alpha x_\beta} + \omega^{\star}(x) \end{aligned} \quad (5.27c)$$

As was the case in Eqs. 5.7a – 5.7c, functions $\omega_i{}^{(\star)}(\mathbf{x})$, $\gamma^{\star}(\mathbf{x})$ and $\omega^{\star}(\mathbf{x})$ are the homogeneous solutions and will not affect our subsequent results, particularly the effective coefficients. Excluding these functions, Eqs. 5.19a – 5.19b contain 18 unknown functions, $N_i{}^{k\alpha}$, $M_i{}^{\alpha}$, $N_i{}^{\alpha}$, G_i, $G_i{}^{(1)}$, $\mathrm{A}_{k\alpha}$, Ξ_α, O_α, Π, $\Pi^{(1)}$, $\Lambda_{k\alpha}$, Z_α, Γ_α, Δ, $\Delta^{(1)}$, $N_i{}^{(1)\alpha\beta}$, $A^{(1)}_{\alpha\beta}$, and $\Lambda_{\alpha\beta}{}^{(1)}$. Their solution is obtained by back substitution of these functions into Eqs. 5.27a – 5.27c and comparing similar terms. We arrive at a set of eighteen unit cell problems the first nine of which have already been determined and are defined in Eqs. 5.8a – 5.10c. The remaining nine local functions, $N_i{}^{(1)\alpha\beta}$, $\Lambda_{\alpha\beta}{}^{(1)}$ and $\mathrm{A}_{\alpha\beta}{}^{(1)}$, satisfy the following unit cell problems:

$$h_\beta{}^{-1} b_{i\beta,\beta y}(\mathbf{y},z) + b_{i3,z}(\mathbf{y},z) = 0 \quad \text{with} \quad b_{ij}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.28a)$$

$$h_\beta{}^{-1} b_{i\beta,\beta y}{}^{(1)}(\mathbf{y},z) + b_{i3,z}{}^{(1)}(\mathbf{y},z) = 0 \quad \text{with} \quad b_{ij}{}^{(1)}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.28b)$$

$$h_\beta{}^{-1} b^{(1)\mu\alpha}_{i\beta,\beta y}(\mathbf{y},z) + b_{i3,z}{}^{(1)\mu\alpha}(\mathbf{y},z) = 0 \quad \text{with} \quad b_{ij}{}^{\mu\alpha}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.28c)$$

$$h_\beta{}^{-1} \gamma_{\beta,\beta y}(\mathbf{y},z) + \gamma_{3,z}(\mathbf{y},z) = 0 \quad \text{with} \quad \gamma_j(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.29a)$$

$$h_\beta{}^{-1} \gamma^{(1)}_{\beta,\beta y}(\mathbf{y},z) + \gamma_{3,z}{}^{(1)}(\mathbf{y},z) = 0 \quad \text{with} \quad \gamma_j{}^{(1)}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.29b)$$

$$h_\beta{}^{-1} a^{(1)\mu\alpha}_{\beta,\beta y}(\mathbf{y},z) + a_{3,z}{}^{(1)\mu\alpha}(\mathbf{y},z) = 0 \quad \text{with} \quad a_i{}^{\mu\alpha}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.29c)$$

$$h_\beta{}^{-1} \tau_{\beta,\beta y}(\mathbf{y},z) + \tau_{3,z}(\mathbf{y},z) = 0 \quad \text{with} \quad \tau_j(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.30a)$$

$$h_\beta{}^{-1} \tau^{(1)}_{\beta,\beta y}(\mathbf{y},z) + \tau_{3,z}{}^{(1)}(\mathbf{y},z) = 0 \quad \text{with} \quad \tau_j{}^{(1)}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.30b)$$

$$h_\beta{}^{-1} \delta^{(1)\mu\alpha}_{\beta,\beta y}(\mathbf{y},z) + \delta_{3,z}{}^{(1)\mu\alpha}(\mathbf{y},z) = 0 \quad \text{with} \quad \delta_i{}^{\mu\alpha}(\mathbf{y},z) N_j{}^{\pm} = 0 \quad \text{on } Z^{\pm} \quad (5.30c)$$

Here, we use the following definitions:

$$b_{ij} = L_{ijm} G_m + M_{ij}\Pi + N_{ij}\Delta - \Theta_{ij} \quad (5.31a)$$

$$b_{ij}{}^{(1)} = L_{ijm} G_m{}^{(1)} + M_{ij}\Pi^{(1)} + N_{ij}\Delta^{(1)} - z\Theta_{ij} \quad (5.31b)$$

$$b_{ij}{}^{(1)k\alpha} = L_{ijm} N_m{}^{(1)k\alpha} + M_{ij} A_{k\alpha}{}^{(1)} - N_{ij}\Lambda_{k\alpha}{}^{(1)} + zC_{ijk\alpha} \quad (5.31c)$$

$$\gamma_j = L_{ji} G_i - M_j \Pi - N_j \Delta - \eta_j \quad (5.32a)$$

$$\gamma_j{}^{(1)} = L_{ji} G_i{}^{(1)} - M_j \Pi^{(1)} - N_j \Delta^{(1)} + z\eta_j \quad (5.32b)$$

$$a_i{}^{(1)k\alpha} = L_{im} N_m{}^{(1)k\alpha} - M_i A_{k\alpha}{}^{(1)} - N_i \Lambda_{k\alpha}{}^{(1)} + zQ_{ik\alpha} \quad (5.32c)$$

$$\tau_j = L_{ji}{}^{\star} G_i - M_j{}^{\star}\Pi - N_j{}^{\star}\Delta + \xi_j \quad (5.33a)$$

$$\tau_j{}^{(1)} = L_{ji}{}^{\star} G_i{}^{(1)} - M_j{}^{\star}\Pi^{(1)} - N_j{}^{\star}\Delta^{(1)} + z\xi_j \quad (5.33b)$$

$$\delta_i{}^{(1)k\alpha} = L_{im}{}^{\star} N_m{}^{(1)k\alpha} - M_i{}^{\star} A_{k\alpha}{}^{(1)} - N_i{}^{\star}\Lambda_{k\alpha}{}^{(1)} + ze_{ik\alpha} \quad (5.33c)$$

The presence of the z coordinate in the unit cell problems 5.28b, 5.28c, 5.29b, 5.29c as well as 5.30b, 5.30c implies that these problems are related to out-of-plane deformation and electric and magnetic field generation in the homogenized plate.

6 Effective properties of homogenized plate and relationships with classical plate

Substitution of Eqs. 5.27a – 5.27c into Eqs. 5.25a – 5.25c gives the leading terms of the asymptotic expansions of mechanical stress, electric displacement and magnetic induction in terms of the local functions obtained via the 18 unit cell problems in Eqs. 5.8a – 5.10c and 5.28a – 5.30c.

$$\sigma_{ij}{}^{(1)} = b_{ij}{}^{\alpha\beta}\omega_{\alpha,x_\beta} - b_{ij}{}^{(0)\alpha\beta}u_{3,x_\alpha x_\beta}{}^{(0)} + b_\beta{}^{ij}\gamma_{,x_\beta} + a_\beta{}^{ij}\omega_{,x_\beta} + b_{ij}T_1{}^{(0)} + b_{ij}{}^{(1)}T_2{}^{(0)} \tag{6.1a}$$

$$D_i{}^{(1)} = \delta_i{}^{\alpha\beta}\omega_{\alpha,x_\beta} - \delta_{ij}{}^{(1)\alpha\beta}u^{(0)}_{3,x_\alpha x_\beta} + \delta_{i\beta}\gamma_{,x_\beta} + \xi_{i\beta}\omega_{,x_\beta} + \tau_i T_1{}^{(0)} + \tau_i{}^{(1)}T_2{}^{(0)} \tag{6.1b}$$

$$B_i{}^{(1)} = \eta_i{}^{\alpha\beta}\omega_{\alpha,x_\beta} - a_i{}^{(1)\alpha\beta}u^{(0)}_{3,x_\alpha x_\beta} + a_{i\beta}\gamma_{,x_\beta} + \gamma_{i\beta}\omega_{\alpha,x_\beta} + \gamma_i T_1{}^{(0)} + \gamma_i{}^{(1)}T_2{}^{(0)} \tag{6.1c}$$

Recalling that $\sigma_{ij}{}^{(0)} = 0$ from Eq. 5.23, then we may use Eq. 4.6 to write down the in-plane force and moment resultants pertaining to the homogenized plate in a form which is reminiscent of the classical composite laminate theory, see for example Gibson [56]. Thus we have:

$$N_{\alpha\beta} = \delta\left\langle b_{\alpha\beta}{}^{\mu\nu}\right\rangle\varepsilon_{\mu\nu} - \delta^2\left\langle b_{\alpha\beta}{}^{(1)\mu\nu}\right\rangle u^{(0)}_{3,x_\mu x_\nu} + \delta\left\langle b_\mu{}^{\alpha\beta}\right\rangle\varphi^\star_{,x_\mu} + \delta\left\langle a_\mu^{\alpha\beta}\right\rangle\psi^\star_{,x_\mu} + \delta^2\left\langle b_{\alpha\beta}\right\rangle T_1{}^{(0)} + \delta^2\left\langle b_{\alpha\beta}{}^{(1)}\right\rangle T_2{}^{(0)} \tag{6.2a}$$

$$M_{\alpha\beta} = \delta^2\left\langle zb_{\alpha\beta}{}^{\mu\nu}\right\rangle\varepsilon_{\mu\nu} - \delta^3\left\langle zb_{\alpha\beta}{}^{(1)\mu\nu}\right\rangle u^{(0)}_{3,x_\mu x_\nu} + \delta^2\left\langle zb_\mu{}^{\alpha\beta}\right\rangle\varphi^\star_{x_\mu} + \delta^2\left\langle za_\mu^{\alpha\beta}\right\rangle\psi^\star_{x_\mu} + \delta^3\left\langle zb_{\alpha\beta}\right\rangle T_1{}^{(0)} + \delta^3\left\langle zb_{\alpha\beta}{}^{(1)}\right\rangle T_2{}^{(0)} \tag{6.2b}$$

To arrive at Eqs. 6.2a and 6.2b we use the following definitions:

$$\psi^\star = \delta\omega, \quad \varphi^\star = \delta\gamma, \quad v_\alpha = \delta\omega_\mu, \quad \varepsilon_{\mu\nu} = \frac{\partial v_\mu}{\partial x_\nu} \tag{6.2c}$$

Likewise, we can write down the averaged electric displacement and magnetic induction by applying the homogenization procedure directly to Eqs. 6.1b and 6.1c. Thus we arrive at:

$$\langle D_\alpha\rangle = \delta\left\langle\delta_\alpha{}^{\mu\nu}\right\rangle\varepsilon_{\mu\nu} - \delta^2\left\langle\delta_\alpha{}^{(1)\mu\nu}\right\rangle u^{(0)}_{3,x_\mu x_\nu} + \delta\left\langle\delta_{\alpha\mu}\right\rangle\varphi^\star_{x_\mu} + \delta\left\langle\xi_{\alpha\mu}\right\rangle\psi^\star_{x_\mu} + \delta^2\left\langle\tau_\alpha\right\rangle T_1{}^{(0)} + \delta^2\left\langle\tau_\alpha{}^{(1)}\right\rangle T_2{}^{(0)} \tag{6.2d}$$

$$\langle B_\alpha\rangle = \delta\left\langle\eta_\alpha{}^{\mu\nu}\right\rangle\varepsilon_{\mu\nu} - \delta^2\left\langle a_\alpha{}^{(1)\mu\nu}\right\rangle u^{(0)}_{3,x_\mu x_\nu} + \delta\left\langle a_{\alpha\mu}\right\rangle\gamma^\star_{x_\mu} + \delta\left\langle\gamma_{\alpha\mu}\right\rangle\omega^\star_{x_\mu} + \delta^2\left\langle\gamma_\alpha\right\rangle T_1{}^{(0)} + \delta^2\left\langle\gamma_\alpha{}^{(1)}\right\rangle\lambda_2{}^{\star(1)} \tag{6.2e}$$

Next, in view of definitions 6.2c, the expressions for the mechanical displacement, Eqs. 5.24a and 5.27a, can be written down as:

$$u_\beta = \left\{v_\beta - x_3 u^{(0)}_{3,\beta x}\right\} + \delta N^{\mu\nu}_\beta\varepsilon_{\mu\nu} - \delta^2 N_\beta{}^{(1)\mu\nu}u^{(0)}_{3,x_\mu x_\nu} + \delta M^\beta_\mu\varphi^\star_{x_\mu} + \delta N^\beta_\mu\psi^\star_{x_\mu} + \delta^2 G_\beta T_1{}^{(0)} + \delta^2 G^{(1)}_\beta T_2{}^{(0)} + \delta^2\omega_\beta{}^\star \tag{6.2f}$$

$$u_3 = \left\{u_3{}^{(0)} + v_3\right\} + \delta N_3{}^{\mu\nu}\varepsilon_{\mu\nu} - \delta^2 N_3{}^{(1)\mu\nu}u^{(0)}_{3,x_\mu x_\nu} + \delta M^3_\mu\varphi^\star_{x_\mu} + \delta N_\mu{}^3\psi^\star_{x_\mu} + \delta^2 G_3 T_1{}^{(0)} + \delta^2 G_3{}^{(1)}T_2{}^{(0)} + \delta^2\omega_3{}^\star \tag{6.2g}$$

Finally, the expressions for the electric and magnetic potentials, Eqs. 5.24b, 5.24c and 5.27b, 5.27c, may be conveniently written down as:

$$\varphi = \delta\varphi^\star + \delta A_{\mu\alpha}\varepsilon_{\mu\alpha} - \delta^2 A^{(1)}_{\alpha\beta}\hat{u}^{(0)}_{3,x_\alpha x_\beta} + \delta\Xi_\alpha\varphi^\star_{x_\alpha} + \delta O_\alpha\psi^\star_{x_\alpha} + \delta^2\Pi T^{(0)}_1 + \delta^2\Pi^{(1)}T^{(0)}_2 + \delta^2\gamma^\star \tag{6.2h}$$

$$\psi = \delta\psi^\star + \delta\Lambda_{\mu\alpha}\varepsilon_{\mu\alpha} - \delta^2\Lambda^{(1)}_{\alpha\beta}\hat{u}^{(0)}_{3,x_\alpha x_\beta} + \delta Z_\alpha\varphi^\star_{x_\alpha} + \delta\Gamma_\alpha\psi^\star_{x_\alpha} + \delta^2\Delta T^{(0)}_1 + \delta^2\Delta^{(1)}T^{(0)}_2 + \delta^2\omega^\star \tag{6.2i}$$

Careful examination of Eqs. 6.2a – 6.2i readily reveals that $\delta\left\langle b^{\mu\nu}_{\alpha\beta}\right\rangle$ are the extensional effective elastic coefficients, $\delta^2\left\langle b^{(1)\mu\nu}_{\alpha\beta}\right\rangle = \delta^2\left\langle zb^{\mu\nu}_{\alpha\beta}\right\rangle$ are the coupling effective elastic coefficients and $\delta^3\left\langle zb^{(1)\mu\nu}_{\alpha\beta}\right\rangle$ are the bending effective elastic coefficients. In fact, the following correspondence is evident, see Gibson [68].

$$A_{11} = \delta\left\langle b_{11}{}^{11}\right\rangle, \quad A_{12} = \delta\left\langle b_{11}{}^{22}\right\rangle, \quad A_{16} = \delta\left\langle b_{11}{}^{12}\right\rangle,$$
$$A_{22} = \delta\left\langle b_{22}{}^{22}\right\rangle, \quad A_{26} = \delta\left\langle b_{22}{}^{12}\right\rangle, \quad A_{66} = \delta\left\langle b_{12}{}^{12}\right\rangle,$$
$$B_{11} = \delta^2\left\langle zb_{11}{}^{11}\right\rangle = \delta^2\left\langle b_{11}{}^{(1)11}\right\rangle,$$
$$B_{12} = \delta^2\left\langle zb_{11}{}^{22}\right\rangle = \delta^2\left\langle b_{11}{}^{(1)22}\right\rangle,$$
$$B_{16} = \delta^2\left\langle zb_{11}{}^{12}\right\rangle = \delta^2\left\langle b_{11}{}^{(1)12}\right\rangle,$$
$$B_{22} = \delta^2\left\langle zb_{22}{}^{22}\right\rangle = \delta^2\left\langle b_{22}{}^{(1)22}\right\rangle,$$
$$B_{26} = \delta^2\left\langle zb_{22}{}^{12}\right\rangle = \delta^2\left\langle b_{12}{}^{(1)22}\right\rangle,$$
$$B_{66} = \delta^2\left\langle zb_{12}{}^{12}\right\rangle = \delta^2\left\langle b^{(1)12}_{12}\right\rangle,$$

$$D_{11} = \delta^3 \left\langle zb_{11}^{(1)11} \right\rangle, \; D_{12} = \delta^3 \left\langle zb_{11}{}^{(1)22} \right\rangle,$$
$$D_{16} = \delta^3 \left\langle zb_{11}{}^{(1)12} \right\rangle,$$
$$D_{22} = \delta^3 \left\langle zb_{22}{}^{(1)22} \right\rangle, \; D_{26} = \delta^3 \left\langle zb_{22}{}^{(1)12} \right\rangle, \quad (6.3)$$
$$D_{66} = \delta^3 \left\langle zb_{12}{}^{(1)12} \right\rangle$$

Furthermore, $\delta \left\langle b_\mu^{\alpha\beta} \right\rangle, \delta \left\langle \delta_\alpha^{\mu\nu} \right\rangle$ are the effective in-plane piezoelectric coefficients, $\delta \left\langle a_\mu^{\alpha\beta} \right\rangle, \delta \left\langle \eta_\alpha^{\mu\nu} \right\rangle$ the effective in-plane piezomagnetic coefficients, $-\delta \left\langle b_{\alpha\beta} \right\rangle$ and $-\left\langle b_{\alpha\beta}^{(1)} \right\rangle$ the effective in-plane thermal expansion coefficients related to the mid-plane temperature variation and the through-the-thickness linear temperature variation, respectively, see Eq. 3.4b, $\delta^2 \left\langle zb_\mu^{\alpha\beta} \right\rangle = \delta^2 \left\langle \delta_\alpha^{(1)\mu\nu} \right\rangle$ are the effective out-of-plane piezoelectric coefficients, $\delta^2 \left\langle za_\mu^{\alpha\beta} \right\rangle = \delta^2 \left\langle a_\alpha^{(1)\mu\nu} \right\rangle$ the effective out-of-plane piezomagnetic coefficients, $-\delta^2 \left\langle zb_{\alpha\beta} \right\rangle$ and $-\delta^2 \left\langle zb_{\alpha\beta}^{(1)} \right\rangle$ the effective out-of-plane thermal expansion coefficients, $-\delta \left\langle \delta_{\alpha\mu} \right\rangle$ and $-\delta \left\langle \gamma_{\alpha\mu} \right\rangle$ are, respectively, the effective dielectric permittivity and magnetic permeability, $-\delta \left\langle \xi_{\alpha\mu} \right\rangle, -\delta \left\langle a_{\alpha\mu} \right\rangle$ are the effective magnetoelectric coefficients, $\delta \left\langle \tau_\alpha \right\rangle$ and $\delta \left\langle \tau_\alpha^{(1)} \right\rangle$ are the effective pyroelectric coefficients, and $\delta \left\langle \gamma_\alpha \right\rangle$ and $\delta \left\langle \gamma_\alpha^{(1)} \right\rangle$ the effective pyromagnetic coefficients. Further, $v_1, \; v_2$ and $u_3^{(0)}$ represent the displacements of the middle plane of the plate and consequently $\varepsilon_{11}, \; \varepsilon_{22}$ and ε_{12} are the mid-surface strains, see Gibson [68]. Similarly, $\frac{\partial^2 u_3^{(0)}}{\partial x_1^2} = \kappa_{x1}$ and $\frac{\partial^2 u_3^{(0)}}{\partial x_2^2} = \kappa_{x2}$ are the bending curvatures associated with bending of the middle surface in the x_1x_3 and x_2x_3 planes, respectively, and $2 \frac{\partial^2 u_3^{(0)}}{\partial x_1 \partial x_2} = k_{xy}$ is the twisting curvature associated with torsion of the middle surface.

It should be pointed out that in the case of the purely elastic case, the results of this model converge exactly to those of Kalamkarov [44], Kalamkarov and Kolpakov [45] and Kalamkarov and Georgiades [65]. In the present work, however, the authors adhere to a completely coupled approach, which results in significantly refined expressions as compared to previously published results, such as those in [65, 66, 71, 72]. All these previously published papers employed a semi-coupled approach, resulting in expressions for the effective coefficients, which do not reflect the influence of all material parameters. For example, the effective elastic coefficients of smart laminates as well as wafer-reinforced plates (as obtained via the semi-coupled approach) given in [65] and [66] depend only on the elastic parameters of the constituents. However, an examination of, say, unit cell problem 5.8a and associated definition 5.11a will reveal that in the completely coupled approach followed in the present work the effective in-plane elastic coefficients are dependent on not only the elastic properties of the constituent materials, but also on the piezoelectric, piezomagnetic, magnetic permeability, dielectric permittivity and other parameters. The same is true for all remaining effective coefficients, as expressions 5.11a – 5.13c and 5.31a – 5.33c reveal. In this sense, the thermoelasticity, piezoelectricity and piezomagnetism problems are entirely coupled, and the solution of one affects the solutions of the others. This feature is captured in the present works, but not in [65, 66, 71, 72]. For the same reasons, if applied to the case of simple laminated structures, the work presented here represents an extension of the classical composite laminate theory (see e.g. [70, 73]) to magneto-piezo-thermo-elastic structures. More important, however, is the fact that the model developed in the present work explicitly allows for different periodicity in the lateral directions. As such, it is readily amenable to the design and analysis of magnetoelectric reinforced plates, such as the wafer-reinforced and rib-reinforced structures shown in the next section. To the authors' best knowledge, this is the first time that completely coupled piezo-magneto-thermo-elastic effective coefficients for reinforced plates are presented and analyzed.

7 Examples of magnetoelectric composite and reinforced plates

The mathematical model developed in Sections 1-6 can be used in analysis and design to tailor the effective elastic, piezoelectric, magnetoelectric and other coefficients of composite and reinforced plates (Figs. 2 and 3) to meet the criteria of specific engineering applications. The main objective of Part II of this work [67] is precisely that.

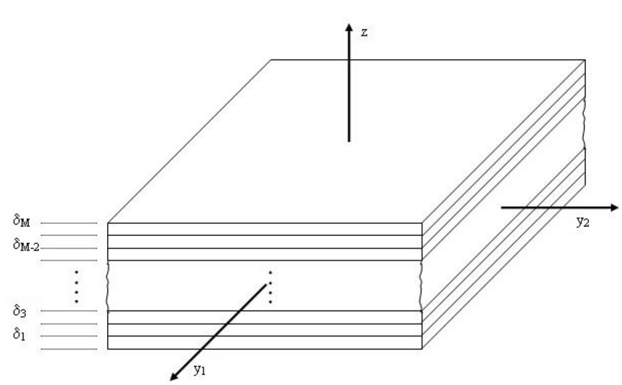
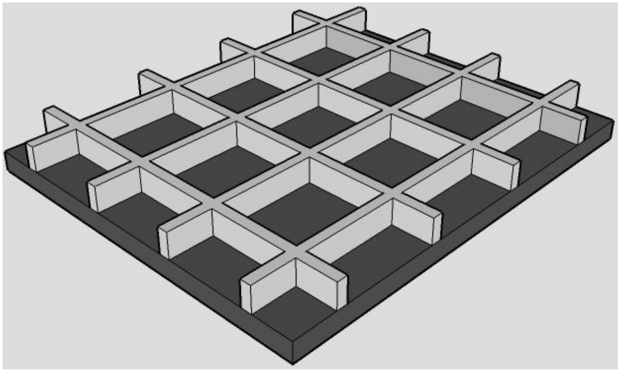
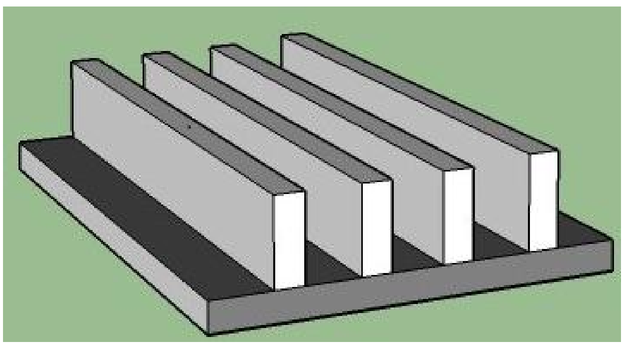


Figure 2: Laminated magnetoelectric composite plate



(a)



(b)

Figure 3: (a) Wafer-reinforced and (b) rib-reinforced magnetoelectric composite plates

At this point, it is worthwhile to reiterate that the nature of the reinforced structures is such that it would be more efficient if we first considered a simpler type of unit cell consisting of only a single reinforcement/actuator. Having dealt with this situation, the effective coefficients of more general structures with multiple families of reinforcements/actuators can readily be determined by superposition. In following this procedure, one must naturally accept the error incurred at the interphase and/or regions of overlap between the actuators/reinforcements or between the matrix and the actuators/reinforcements. However, our approximation is quite accurate, since these regions are highly localized and do not contribute significantly to the integral over the entire unit cell domain. Essentially, the error incurred will be negligible if the dimensions of the actuators/reinforcement are much smaller than the spacing between them. We note for example the asymptotic homogenization model developed by Kalamkarov [44] for the purely elastic case of thin composite plates reinforced with mutually perpendicular ribs. In that work, he determined that if the spacing between the unit cells is at least ten times bigger than the thickness of the reinforcements the error in the values of the effective elastic coefficients incurred by ignoring the regions of overlap (between the reinforcements) is less than 1%. A complete mathematical justification for this argument in the form of the so-called principle of the split homogenized operator has been provided by Bakhvalov and Panasenko [42].

Furthermore, we note that the general purely mathematical aspects of the asymptotic homogenisation procedure can be found in [41, 42]. The pertinent mathematical details for the asymptotic homogenization of thin-walled inhomogeneous structures can be found in [56]-[59]. The convergence of the two-scale asymptotic method is proven in these papers on the basis of G-convergence. These purely mathematical aspects of asymptotic homogenization are beyond the scope of our papers, which are aimed at developing micromechanical models and deriving results for effective properties of magnetoelectric and other smart structures of practical importance.

8 Summary and concluding remarks

The method of asymptotic homogenization is used to analyze a periodic smart composite plate of rapidly varying thickness with elastic, piezoelectric and piezomagnetic constituents. A set of eighteen fully-coupled three-dimensional local unit cell problems is derived. However, unlike classical homogenization schemes, the derived unit cell problems are shown to depend on boundary conditions rather than periodicity in the transverse direction. The solution of the unit cell problems yields a set of functions which, when averaged over the volume of the periodicity cell, can be used to determine the effective elastic, piezoelectric, dielectric permittivity and other coefficients of the homogenized anisotropic smart plate. Of interest among these coefficients are the so-called product coefficients, which are present in the behavior of the macroscopic composite as a result of the interactions between the various phases but can be absent from the constitutive behavior of the individual phases of the composite material. The effective coefficients are substituted into the governing equations of the structure, which in turn yield a set of local functions. These functions allow us to make very accurate predictions concerning the three-dimensional local structure of the mechanical stress and displacement fields, electric and magnetic potentials etc.

The local problems are expressed in a form that shows that they are completely determined by the geometrical and material characteristics of the unit cell of the smart plate and are totally independent of the global formulation of the original problem. It follows that derived effective coefficients are universal in nature and may be used to analyze different types of boundary value problems associated with a given smart structure. Finally, it is shown that in the limiting case of a thin elastic plate of uniform thickness the derived model converges to the familiar classical plate model.

Appropriately, in part II of this work, Hadjiloizi et al. [67], illustrate the theory developed here using the practically important examples of magnetoelectric thin laminates and magnetoelectric wafer-reinforced composite plates. In both cases it is shown that the developed model can be used to tailor the properties of a given structure to conform to the requirements of a particular engineering application by changing appropriate geometrical or material parameters.

Acknowledgement: The authors would like to acknowledge the financial support of the Cyprus University of Technology (1st, 3rd and 4th authors), the Research Unit for Nanostructured Materials Systems (1st, 3rd and 4th authors) and the Natural Sciences and Engineering Research Council of Canada (2nd author).

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